# SOME PROPERTIES OF HARMONIC MAPPINGS

by

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## CONTENTS

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# INTRODUCTION

A harmonic map between Riemannian manifolds satisfies, in local coordinates, a second order semi-linear elliptic system of equations. This system of equations arise as the Euler-Lagrange equations of a natural Dirichlet or energy integral on maps between manifolds, which directly generalises the classical Dirichlet integral. Particular interest in harmonic maps has thus sprung up in connection with the problem of minimal surfaces in Riemannian manifolds.

This thesis begins with a brief introduction to harmonic maps, putting the concepts into a general framework and recording certain basic but important properties of harmonic maps. The second chapter is founded on a remark of Eells and Sampson [8] that a holomorphic map between Kähler manifolds is harmonic. Here a calculation is made of the Laplacian of a decomposed energy density and application of it is made in the holomorphic case. The formula obtained is used in conjunction with harmonic map methods to give a greatly simplified proof of a theorem of Cheng characterising the Euclidean sphere by the eigenfunctions of its Laplacian.

Up until the beginning of the writing of this thesis hardly anything was known about harmonic maps from non-compact domain. Chapter three deals with two situations, one ensuring that the energy density is bounded and another ensuring the total energy is infinite, some contrasts are given including a counter-example to a tempting conjecture. While some of these results rely on curvature restrictions a separate chapter has been reserved for this topic and among those areas considered are maps from manifolds with boundary, a classification problem for maps of small energy and a few brief remarks about the second variation.

Chapter five is a discussion of an old paper of Shibata concerning the existence of harmonic diffeomorphisms of surfaces in which many mistakes have been found. Many of these are corrected but the final solution is not found and an alternative approach to the problem is proposed. A short appendix is attached in which the connection between certain harmonic and holomorphic maps is pointed out. This is viewed as a special case in which an equidistribution theory for harmonic maps actually exists, nothing of this nature is known in general.

# CHAPTER 1

# HARMONIC MAPS

## (a) Basic notions.

Throughout this work, unless stated to the contrary, $(M,g)$ and $(N,h)$ will denote two $C^\infty$ Riemannian manifolds, of dimensions $m$ and $n$ respectively, which are assumed to be connected and without boundary.

If $\pi : W \to M$ is a $C^\infty$ vector bundle on $M$ the space of smooth ($C^\infty$) sections will be denoted by $A(W)$. A connection $\nabla^W$ on $W$ and a fibre metric $k$ define a Riemannian structure on $W$ if $\nabla k = 0$, writing $\langle \phi, \psi \rangle$ for $k(\phi,\psi)$ this means

$$X\langle \phi, \psi \rangle = \langle \nabla^W_X \phi, \psi \rangle + \langle \phi, \nabla^W_X \psi \rangle$$

where $X \in A(TM)$ is a vector field on $M$ and $\phi, \psi \in A(W)$.

The curvature of $\nabla^W$ is given by

$$R^W(X,Y)\phi = -\nabla^W_X \nabla^W_Y \phi + \nabla^W_Y \nabla^W_X \phi + \nabla^W_{[X,Y]}\phi = -R^W(Y,X)\phi \text{ and in case}$$

$W = TM$ define $R(X,Y,Z,U) = \langle R(X,Y)Z, U \rangle$.

If $u, v \in T_xM$ are orthogonal and of unit length the sectional curvature determined by $u, v$ is

$$K(u \wedge v) = R(u,v,u,v).$$

An expression like $A \le \mathrm{Riem}^M \le B$ will mean that all sectional curvatures of $M$ lie within the indicated bounds.

Example.

Suppose $\phi : (M,g) \to (N,h)$ is smooth and let $\nabla$ denote the Levi-Civita connection of $N$ then the pull back via $\phi$ of $TN$, denoted $\phi^*TN$ inherits a Riemannian structure the connection of which will be denoted $\nabla^\phi$.

The elements of $A(\Lambda^p T^*M \otimes W)$ are called W-valued p-forms and will also be denoted by $A^p(W)$. There are defined on $A^p(W)$ various operators. The exterior derivative $d: A^p(W) \to A^{p+1}(W)$ is given by

$$d\omega(X_1,\ldots,X_{p+1}) = \sum_{i=1}^{p+1} (-1)^{i+1} \nabla_{X_i} \omega(X_1,\ldots,\hat{X}_i,\ldots,X_{p+1})$$

$$+ \sum_{i<j} (-1)^{i+j} \omega([X_i,X_j],X_1,\ldots,\hat{X}_i,\ldots,\hat{X}_j,\ldots X_{p+1})$$

where $X_i \in A(TM)$ $i = 1,\ldots,p+1$, and the covariant derivative by

$$(\nabla_Y\omega)(X_1,\ldots,X_p) = \nabla_Y\omega(X_1,\ldots,X_p) - \sum_{i=1}^{p} \omega(X_1,\ldots,\nabla^M_Y X_i,\ldots,X_p)$$

for $X_1,\ldots,X_p$, $Y \in A(TM)$, $\omega \in A^p(W)$.

Note that for a 1-form $\omega, d\omega(X,Y) = (\nabla_X\omega)(Y) - (\nabla_Y\omega)(X)$. For $\omega$, $\theta \in A^p(W)$ let $\langle\omega,\theta\rangle$ denote the smooth function whose value at $x \in M$ is given by

$$\langle\omega,\theta\rangle_x = \sum_{i_1<\ldots<i_p} \langle\omega(e_{i_1},\ldots e_{i_p}),\ \theta(e_{i_1},\ldots,e_{i_p})\rangle$$

where $\{e_1,\ldots,e_m\}$ is an orthonormal basis of $T_xM$, the tangent space to M at x.

The codifferential $d^*: A^p(W) \to A^{p-1}(W)$ is given by

$$(d^*\omega)_x(u_1,\ldots,u_{p-1}) = - \sum_{i=1}^{m} (\nabla_{e_i}\omega)_x(e_i,u_1,\ldots,u_{p-1})$$

$u_1,\ldots,u_{p-1} \in T_xM$ and as above $\{e_1,\ldots,e_m\}$ is an orthonormal basis of $T_xM$, it satisfies

$$\int_M \langle d^*\omega,\theta\rangle dv_g = \int_M \langle\omega,d\theta\rangle dv_g$$

for all $\omega \in A^{p+1}(W)$, $\theta \in A^p(W)$ of compact support. (Here $dv_g$ is the Riemannian volume element of $(M,g)$).

The associated Laplacian, given by $\Delta = dd^* + d^*d$, is a self adjoint strongly elliptic operator, $\omega \in A^p(W)$ is a harmonic W-valued p-form if $\Delta\omega = 0$.

(b) Harmonic maps.

Suppose now that $\phi:(M,g) \to (N,h)$ is smooth then the differential, $d\phi$, of $\phi$ can be regarded as a 1-form with values in $\phi^*TN$.

1.1. DEFINITION.

The *energy density* of $\phi$ is the function given by

$$e(\phi) = \tfrac{1}{2}\langle d\phi, d\phi\rangle, = \tfrac{1}{2}\ \mathrm{Trace}\ (\phi^*h).$$

If $M'$ is a relatively compact domain of $M$, $M' \subset\subset M$, the *energy of* $\phi$ *over* $M'$ is

$$E(\phi,M') = \int_{M'} e(\phi)dv_g\ .$$

In case $M' = M$ reference to it will be dropped and the energy simply denoted by $E(\phi)$. $E$ thus defines a positive real valued function on maps from $M$ to $N$.

1.2. DEFINITION

A $C^2$ map $\phi:(M,g) \to (N,h)$ is *harmonic* on $M' \subset\subset M$ if it is a critical point of $E(\ ,M')$ with respect to all variations supported in $M'$, $\phi$ is harmonic on $M$ if it is harmonic on $M'$ for all $M' \subset\subset M$.

1.3. PROPOSITION. [8]

The $C^2$ map $\phi$ is harmonic if and only if its differential $d\phi \in A^1(\phi^*TN)$ satisfies the equation $d^*d\phi = \text{Trace } \nabla^\phi d\phi = 0$.

1.4. DEFINITION.

For any $C^2$ map $\phi:M \to N$ the quantity $-d^*d\phi$ is called the *tension field of* $\phi$ and is written $\tau(\phi)$. Thus a map is harmonic if and only if its tension field vanishes, $\tau(\phi) = 0$.

(c) Regularity

While the concept of energy has been defined only for smooth maps it is clear that it can be defined for maps which have distributional derivatives which are locally square integrable. This leads to the question of whether such a map which is also a critical point of the energy does in fact satisfy a meaningful equation $\tau(\phi) = 0$. The best result in this direction is due to Hildebrandt.

1.5. PROPOSITION

Suppose $\phi:(M,g) \to (N,h)$ is continuous and possesses square integrable distributional derivatives. If $\phi$ is a critical point of the energy $E(\ ,M')$ for all $M' \subset\subset M$ then $\phi$ is smooth ($C^\infty$) and satisfies the equation $\tau(\phi) = 0$.

(d) Compositions.

It is not true in general that the composition of two harmonic maps is harmonic.

1.6. PROPOSITION [ 8 ]

Suppose $\phi:M \to N$, $\psi:N \to P$ are maps of Riemannian manifolds then

$$\tau(\psi \circ \phi) = d\psi(\tau(\phi)) + \text{Trace } \nabla d\psi(d\phi, d\phi).$$

1.7. DEFINITION

A map $\phi:(M,g) \to (N,h)$ is called *totally geodesic* if it satisfies the identity $\nabla^{\phi} d\phi = 0$. Such maps carry geodesics of M linearly into geodesics of N.

* For maps from product manifolds it is a simple calculation to show the following holds.

1.9 PROPOSITION

If $\phi:M^1 \times M^2 \to M$ is a map from a Riemannian product into some Riemannian manifold then

$$\tau(\phi) = \tau^1(\phi) + \tau^2(\phi)$$

where $\tau^1(\phi), \tau^2(\phi)$ have the obvious meanings.

Finally, for maps into submanifolds.

1.10 PROPOSITION [8]

Suppose $N \hookrightarrow P$ is a submanifold and $\phi:M \to N$. If $\Phi$ denotes the composition $\Phi:M \xrightarrow{\phi} N \hookrightarrow P$, then $\phi$ is harmonic $\Leftrightarrow \tau(\Phi) \perp N$.

* 1.8. PROPOSITION [8]

If $\phi: M \to N$ is harmonic and $\psi: N \to P$ is totally geodesic then $\psi \circ \phi : M \to P$ is harmonic.

# CHAPTER 2

# THE COMPLEX CASE.

## (a) Complex bundles

In order to fix notation and conventions a brief survey of complex bundle theory is included.

### 2.1. DEFINITION

Suppose $\pi : W \to M$ is a vector bundle of even fibre dimension. A *complex structure* S on W is a smooth section $J$ of $W^* \otimes W \to M$ ($W^*$ = dual bundle of W) which when regarded as an automorphism of the fibres satisfies $J^2 = -\mathrm{id}$. Each fibre becomes a complex vector space via the definition of multiplication by $i = \sqrt{-1}$ as $iu = Ju$ for $u \in W$. If $W = TM$ then J is called an almost complex structure on M and (M,J) is an almost complex manifold.

### 2.2. DEFINITION

Suppose $(M, J^M)$, $(N, J^N)$ are almost complex manifolds. A differentiable map $\phi : M \to N$ is *holomorphic* if $d\phi \circ J^M = J^N \circ d\phi$, and *anti-holomorphic* if $d\phi \circ J^M = -J^N \circ d\phi$. A statement like "$\phi$ is $\pm$ holomorphic" will mean that $\phi$ is either holomorphic or anti-holomorphic.

For example, the almost complex structure on $\mathbb{C}^m$ with coordinates $z^1, \ldots, z^m$, $z^\alpha = x^\alpha + iy^\alpha$ is given by

$$J(\partial/\partial x^\alpha) = \partial/\partial y^\alpha, \quad J(\partial/\partial y^\alpha) = -\,\partial/\partial x^\alpha$$

and the equation $d\phi \circ J = J \circ d\phi$ for a map $\phi : \mathbb{C} \to \mathbb{C}$ is nothing other than the Cauchy-Riemann equations.

A holomorphic chart on $(M,J)$ is a locally defined holomorphic diffeomorphism onto some open set in $\mathbb{C}^m$, if every point of $M$ is contained in the domain of definition of just such a chart, then $J$ is said to be integrable and $(M,J)$ is called a complex manifold.

2.3. DEFINITION

A *Hermitian metric* $k$ on a complex bundle $(W,J)$ is a fibre metric on $W$ for which $J$ is an isometry at each point of $M$. Thus if $u,v \in W_p$, the fibre of $W$ over $p \in M$,

$$k(u,v) = k(Ju,Jv).$$

Set $W^{\mathbb{C}} = W \otimes_{\mathbb{R}} \mathbb{C}$ and extend $J$ to act on $W^{\mathbb{C}}$ by complex linearity and extend a by complex bilinearity. The eigenvalues of $J$ are $\pm i$ and the respective eigenspaces are denoted $W^{1,0}$, $W^{0,1}$ so that $W^{\mathbb{C}} = W^{1,0} \oplus W^{0,1}$. If a metric is defined on $W^{\mathbb{C}}$ by $\langle u,v\rangle = k(u,\bar{v})$ this decomposition becomes orthogonal. It should be noted that $W$, $W^{1,0}$, $W^{0,1}$ are all isomorphic as real bundles via the maps

$$\begin{array}{ccccc} W & \to & W^{1,0} & \to & W^{0,1} \\ u & \mapsto & \frac{1}{\sqrt{2}}(u-iJu) & \mapsto & \frac{1}{\sqrt{2}}(u+iJu) \end{array}$$

which are in fact isometric in the fibres. The correspondence on the right is called complex conjugation and in future will be denoted by $v \to \bar{v}$.

In the case W = TM there is induced in T*M a complex structure, still called J, whose $\pm i$ eigenspaces in $T^*_{\mathbb{C}}M = T^*M \otimes_{\mathbb{R}} \mathbb{C}$ are denoted by $T^*_{1,0}M$, $T^*_{0,1}M$. If (M,J) is a complex manifold with some local coordinate $z^1,\dots,z^m$ a basis of $T^*_{1,0}M$ is given by $dz^1,\dots,dz^m$ and a basis of $T^*_{0,1}M$ by $d\bar{z}^1,\dots,d\bar{z}^m$. A differential form on M is of type (p,q) if it is a sum of expressions of the form

$$\phi(z)\, dz^{i_1} \wedge \dots \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \wedge \dots \wedge d\bar{z}^{j_q}.$$

The collection of smooth differential forms of type (p,q) is denoted by $A^{p,q}(M)$. For a complex vector bundle W over M there is similarly defined $A^{p,q}(W)$, the space of W-valued (p,q) forms.

(b) Operators on complex bundles

The ideas of connection, exterior derivative etc. for real bundles have their analogues in complex bundles.

2.4. DEFINITION

Suppose that (W,J,k) is a complex bundle together with a Hermitian metric. A connection $\nabla$ on (W,J,k) will be called *Hermitian* if

$$X.\langle\phi,\psi\rangle = \langle\nabla_X\phi,\psi\rangle + \langle\phi,\nabla_{\bar{X}}\psi\rangle$$

for $X \in A(T^{\mathbb{C}}M)$, $\phi, \psi \in A(W)$.

2.5. Example

If (TM,J,g) together with its Levi-Civita connection is under consideration then the complex bundle $T^{1,0}M$ becomes Hermitian connected via the complexification of $\nabla$ and the inner product $\langle u,v\rangle = g(u,\bar{v})$. This is because if $X \in A(T^{\mathbb{C}}M)$ $Y,Z \in A(T^{1,0}M)$ then

$$X\langle Y,Z\rangle = Xg(Y,\bar{Z}) = g(\nabla_X Y,\bar{Z}) + g(Y,\nabla_X \bar{Z})$$

$$= \langle\nabla_X Y,Z\rangle + \langle Y,\nabla_{\bar{X}} Z\rangle.$$

Similarly for the bundle $\phi^* T^{\mathbb{C}}N$ if $\phi: M \to N$.

In the example just cited there can occur the condition $\nabla J = 0$ which ensures that the Riemannian geometry and the complex structure are intimately related. Such manifolds have a special name.

2.6. DEFINITION

Suppose $(M,J,g)$ is a complex manifold with Hermitian metric, if $\nabla J = 0$ with respect to the Levi-Civita connection, then the metric is called a *Kähler metric* and $(M,J,g)$ is called a *Kähler manifold*.

If $(M,J,g)$ is Kähler and $u \in TM$ has unit length, the *holomorphic sectional curvature determined by* u is the quantity

$$H(u) = R(u,Ju,u,Ju)$$

where R is the usual Riemann curvature. For $u,v \in T_xM$ of unit length, the *holomorphic bisectional curvature determined by* u *and* v is given by

$$H(u,v) = R(u,Ju,v,Jv).$$

An expression like $\mathrm{HRiem}^M < A$ will mean that all holomorphic sectional curvatures are less than A, similarly $\mathrm{HBRiem}^M$ refers to the holomorphic bisectional curvatures.

The complex forms of these notions will be useful later, so first extend $g$ to be complex bilinear over $T^{\mathbb{C}}M$ and extend $\nabla$ to complex vector fields then:

2.7. LEMMA

If M is Kähler and $u', v' \in T_x^{1,0}M$ are of unit length then the expression $R(\bar{u}', v', u', \bar{v}')$ is the holomorphic bisectional curvature determined by the real parts of $\sqrt{2}u'$ and $\sqrt{2}v'$.

Proof

According to paragrpah (a) $u'$ and $v'$ can be written

$$u' = 1/\sqrt{2}(u - iJu), \quad v' = 1/\sqrt{2}(v - iJv) \text{ with } u,v \in T_xM$$

both of unit length. Using the symmetry properties of R together with the identity $R(W,X,Y,Z) = R(JW,JX,Y,Z)$ for all $W,X,Y,Z \in A(TM)$ on a Kähler manifold (see [16,II]), the identity

$$R(\bar{u}',v',u',\bar{v}') = R(u,v,u,v) + R(u,Jv,u,Jv) \qquad (1)$$

is easy to deduce. Now Bianchi's identity shows that

$$R(u,Ju,v,Jv) + R(u,v,Jv,Ju) + R(u,Jv,Ju,v) = 0.$$

Rewriting this, $R(u,v,Ju,Jv) - R(u,Jv,Ju,v) = R(u,Ju,v,Jv)$ and applying J, $R(u,v,u,v) + R(u,Jv,u,Jv) = R(u,Ju,v,Jv)$ which, together with (1), furnishes the result.

2.8. COROLLARY

Under the same hypotheses on $u'$ the expression $R(\bar{u}',u',u',\bar{u}')$ is the holomorphic sectional curvature determined by the real part of $\sqrt{2}u'$

Proof

Just put $u' = v'$ in 2.7.

Suppose now $(M,J)$ is a complex manifold, the complex structure of $M$ induces a decomposition of the exterior derivative on $W$ valued forms according to the rule; if $\omega \in A^{p,q}(W)$

$$d'\omega = (p+1,q) - \text{part of } d\omega$$
$$d''\omega = (p,q+1) - \text{part of } d\omega$$

so that $d\omega = d'\omega + d''\omega$.

Similarly, the codifferential becomes

$$d^* = d^{*\prime} + d^{*\prime\prime} .$$

The complex Laplacians are given by

$$\Delta' = d'd^{*\prime} + d^{*\prime}d'$$
$$\Delta'' = d''d^{*\prime\prime} + d^{*\prime\prime}d''$$

The following lemma will be of use.

2.9. LEMMA [34]

In the case that $W$ is the trivial complex line bundle with its usual flat connection and $(M,J,g)$ is Kähler the relation

$\Delta' = \Delta'' = \frac{1}{2}\Delta$ holds.

That this is not the case for $W$ an arbitrary holomorphic bundle is shown in [ 5 ].

(c) A decomposition formula

In this section $M,N$ will be complex manifolds whose complex structures will both be denoted by $J$ (it should be clear which manifold is being referred to at any point). Let $\phi:M \to N$ be a map and extend its derivative by complex linearity to a map $d^{\mathbb{C}}\phi:T^{\mathbb{C}}M \to T^{\mathbb{C}}N$. This map has a decomposition as follows:

$$d'\phi : T^{1,0}M \to T^{\mathbb{C}}M \xrightarrow{d^{\mathbb{C}}\phi} T^{\mathbb{C}}N \to T^{1,0}N$$

$$d''\phi : T^{0,1}M \to T^{\mathbb{C}}M \xrightarrow{d^{\mathbb{C}}\phi} T^{\mathbb{C}}N \to T^{0,1}N$$

where the two maps on the left are inclusions and those on the right are the natural projections onto the appropriate spaces. Thus $d'\phi \in A^{1,0}(\phi^*T^{1,0}N)$ and $d''\phi \in A^{1,0}(\phi^*T^{0,1}N)$, finally set $e'(\phi) = |d'\phi|^2$, $e''(\phi) = |d''\phi|^2$ so that $e(\phi) = e'(\phi) + e''(\phi)$. It should be noted that $\phi$ is holomorphic if and only if $e''(\phi) = 0$ and anti-holomorphic if and only if $e'(\phi) = 0$. The task now is to calculate $\Delta e'(\phi)$ and $\Delta e''(\phi)$.

2.10. THEOREM [cf. 2]

Suppose $(M,g)$ is a Kähler manifold and that $W$ is a Hermitian connected bundle over $M$. If $\omega \in A^{1,0}(W)$ satisfies $d\omega = 0$, $d^*\omega = 0$ then

$$\tfrac{1}{2}\Delta|\omega|^2 = |\nabla\omega|^2 + \langle S(\omega),\omega\rangle$$

at each point of $M$. Here $S(\omega)$ is given at $p \in M$ by

$$S(\omega)(u) = R^W(u,\bar{e}_s)\omega(e_s) - \omega(R^M(u,\bar{e}_s)e_s)$$

where $(e_1,\ldots,e_m)$ is an orthonormal basis of $T_p^{1,0}M$, and the summation convention has been used.

Proof

The idea of the proof is to derive an expression for $\Delta'\omega$ and $\Delta''\omega$ in terms of curvature. To this end let $p \in M$ and using the Kähler structure of $M$ find a holomorphic normal coordinate system $(z^1,\dots,z^m)$ centred at p. Let $E_j$ be the vector field given in these coordinates by $E_j = \partial/\partial z_j$ Then

(i) $\{E_j\}_{j=1}^m$ is othonormal at p

(ii) $\nabla_{E_i} E_j = 0$ at p

(iii) $[E_i, E_j] = 0$ in their domain of definition.

(iv) $\nabla_{\bar{E}_i} E_j = \nabla_{E_i} \bar{E}_j = 0$ throughout their domain of definition.

Finally, let $e_j$ denote the value of $E_j$ at p.

To begin, $(d^{*\prime}d'\omega)_p(e_i) = -(\nabla_{\bar{e}_s} d'\omega)(e_s, e_i)$ (summation convention)

$$= -\nabla^W_{\bar{e}_s} d'\omega(E_s, E_i) \quad \text{by (ii) and (iv)}$$

$$= -\nabla^W_{\bar{e}_s}((\nabla_{E_s}\omega)(E_i) - (\nabla_{E_i}\omega)(E_s))$$

$$= -(\nabla_{\bar{E}_s} \nabla_{E_s}\omega)(e_i) + (\nabla_{\bar{E}_s} \nabla_{E_i}\omega)(e_s) \text{ by (ii).}$$

On the other hand, set $g_{s\bar{t}} = g(E_s, \bar{E}_t)$ and let $g^{s\bar{t}}$ denote its inverse matrix. Then

$$d^{*\prime}\omega = -g^{s\bar{t}}(\nabla_{\bar{E}_t}\omega)(E_s) \text{ in a neighbourhood of p and}$$

$$(d'd^{*\prime}\omega)_p(e_i) = -\nabla^W_{e_i}(g^{s\bar{t}}(\nabla_{\bar{E}_t}\omega)(E_s))$$

$$= -(\nabla_{E_i}\nabla_{\bar{E}_s}\omega)(e_s) \quad \text{by (ii)}$$

and because $e_i(g^{s\bar{t}}) = 0$ and $g^{s\bar{t}} = \delta_{st}$ at p by (i).

Hence, $\Delta'\omega(e_i) = -(\nabla_{\bar{E}_s}\nabla_{E_s}\omega)(e_i) - ((\nabla_{E_i}\nabla_{\bar{E}_s}\omega) - (\nabla_{\bar{E}_s}\nabla_{E_i}\omega))(e_s)$

but (iii) shows that this can be written

$$\Delta'\omega(e_i) = -(\nabla_{\bar{E}_s}\nabla_{E_s}\omega)(e_i) + R^W(e_i,\bar{e}_s)\omega(e_s) - \omega(R^M(e_i,\bar{e}_s)e_s). \qquad (1)$$

In a similar vein

$$\begin{aligned}(d^{*''}d''\omega)(e_i) &= -(\nabla_{e_s}d''\omega)(\bar{e}_s,e_i)\\ &= -\nabla^W_{e_s}d''\omega(\bar{E}_s,E_i) \qquad \text{by (ii) and (iv)}\\ &= -\nabla^W_{e_s}((\nabla^W_{\bar{E}_s}\omega)(E_i) - (\nabla_{E_i}\omega)(\bar{E}_s)) \text{ by (ii),(iii) and (iv)}\\ &= -(\nabla_{E_s}\nabla_{\bar{E}_s}\omega)(e_i) \qquad \text{by (ii) and because } \omega \text{ is of type } (1,0)\end{aligned}$$

while $d^{*''}\omega$ is of type $(1,-1)$ which is just another way of saying it is zero. Hence $d''d^{*''}\omega = 0$ and

$$\Delta''\omega(e_i) = (\nabla_{e_s}\nabla_{\bar{E}_s}\omega)(e_i). \qquad (2)$$

The second step of the proof is to write, according to 2.9,

$$\begin{aligned}\tfrac{1}{2}\Delta|\omega|^2 &= \nabla_{\bar{e}_s}E_s|\omega|^2\\ &= \langle\nabla_{\bar{e}_s}\nabla_{E_s}\omega,\omega\rangle + \langle\omega,\nabla_{e_s}\nabla_{\bar{E}_s}\omega\rangle + \langle\nabla_{E_s}\omega,\nabla_{E_s}\omega\rangle\\ &\qquad + \langle\nabla_{\bar{E}_s}\omega,\nabla_{\bar{E}_s}\omega\rangle\\ &= -\langle\Delta'\omega,\omega\rangle - \langle\omega,\Delta''\omega\rangle + |\nabla\omega|^2 + \langle\mathbf{S}(\omega),\omega\rangle \qquad (3)\end{aligned}$$

using (1) and (2).

The hypotheses $d\omega = 0$, $d^*\omega = 0$ imply that $\Delta'\omega = \Delta''\omega = 0$ thus $\frac{1}{2}\Delta|\omega|^2 = |\nabla\omega|^2 + \langle S(\omega)\omega\rangle$ at p, but p was chosen arbitrarily so the theorem is proved.

2.11. COROLLARY [20]

Suppose $\phi:M \to N$ is a harmonic map of Kähler manifolds then, keeping the notation of the theorem,

$$\tfrac{1}{2}\Delta e'(\phi) = \tfrac{1}{2}|\nabla d'\phi|^2 + \langle d'\phi(R^M(\bar{e}_s,e_i)e_s),d'\phi(e_i)\rangle$$
$$- R^N(d\phi(\bar{e}_s),d\phi(e_i),d'\phi(e_s),\overline{d'\phi(e_i)}) \qquad (4)$$

and $$\tfrac{1}{2}\Delta e''(\phi) = \tfrac{1}{2}|\nabla d''\phi|^2 + \langle d''\phi(R^M(\bar{e}_s,e_i)e_s,d''\phi(e_i)\rangle$$
$$- R^N(d\phi(\bar{e}_s),d\phi(e_i),d''\phi(e_s),\overline{d''\phi(e_i)}) \qquad (5)$$

Proof

As was noted in 1.3 the equation for a harmonic map is $d^*d\phi = 0$, so all the more so $d^*d'\phi = d^*d''\phi = 0$, while a local calculation shows that $dd\phi = 0$ and thus $dd'\phi = dd''\phi = 0$. Consequently $d'\phi \in A^{1,0}(\phi^*T^{1,0}N)$ and $d''\phi \in A^{1,0}(\phi^*T^{0,1}N)$ satisfy the conditions of the theorem and the corollary can be read off immediately.

These formulae are particularly useful in the case that $\phi$ is $\pm$ holomorphic for then they simplify further.

2.12. PROPOSITION

If $\phi:M \to N$ is a holomorphic mapping between Kähler manifolds then

$$\Delta e(\phi) = \tfrac{1}{2}|\nabla d\phi|^2 + \mathrm{Ricci}^M(d\phi, d\phi) - R^N(\overline{d\phi(e_s)}, d\phi(e_i), d\phi(e_s), \overline{d\phi(e_i)})$$

and if $\mathrm{Ricci}^M \geq 0$ and $\mathrm{HBRiem}^N \leq 0$ then $e(\phi)$ is subharmonic. If $M$ has complex dimension 1 then $\mathrm{HBRiem}^N \leq 0$ may be replaced by $\mathrm{HRiem}^N \leq 0$.

Proof.

Since $\phi$ is a holomorphic map between Kähler manifolds it is harmonic [ 8 ] and also $e''(\phi) = 0$. Thus $e'(\phi)=e(\phi)$ and henceforth the primes will be dropped.

Because $\phi$ is holomorphic, $d\phi$ preserves the type of vectors i.e.

$$d\phi(e_i) = d'\phi(e_i)$$

$$\text{and} \quad d\phi(\bar{e}_s) = \overline{d\phi(e_s)} \quad .$$

Putting together these facts with the formula for $\Delta e'(\phi)$ gives

$$\tfrac{1}{2}\Delta e(\phi) = |\nabla d\phi|^2 + \langle d\phi(R^M(\bar{e}_s, e_i)e_s, d\phi(e_i)\rangle$$

$$- R^N(\overline{d\phi(e_s)}, d\phi(e_i), d\phi(e_s), \overline{d\phi(e_i)}). \tag{6}$$

The last term is a sum of terms of the form $-R^N(\bar{u}', v', u', \bar{v}')$ and by 2.7 and hypothesis is non-negative.

If $M$ has complex dimension 1 there is only one term with $u' = v'$ then 2.8 applies. As for the second term:

Claim

If $A \leq \mathrm{Ricci}^M \leq B$, i.e. all eigenvalues of the Ricci transformation of $M$ lies between these limits then

$$Ae(\phi) \le \langle d\phi(R^M(\bar{e}_s,e_i)e_s, d\phi(e_i)\rangle \le Be(\phi).$$

Which fact concludes the proof that $\Delta e(\phi) \ge 0$.

There remains to prove the claim. To this end let $\{E_i\}_{i=1}^m$ now denote an orthonormal (1,0)frame field in a neighbourhood of p and let $\{\omega^i\}_{i=1}^m$ be the dual coframe field. Set $d\phi = \phi_k\,\omega^k$ so that $\phi_k$ is a locally defined section of $\phi^*T^{1,0}N$. Also set

$$R^M(\bar{e}_s,e_i)e_s = R_{\bar{s}is}{}^{\ell}E_\ell \text{ so that } \mathrm{Ricci}^M(\bar{E}_i,E_j) = R_{\bar{i}sj}{}^{s} = R_{\bar{i}j}\ .$$

Thus $\langle d\phi(R^M(\bar{e}_s,e_i)e_s, d\phi(e_i)\rangle = \langle \phi_\ell\ R_{\bar{s}is}{}^{\ell}\ ,\ \phi_i\rangle$

$$= \langle \phi_\ell\ R_{\bar{s}is\bar{\ell}}\ ,\phi_i\rangle$$

$$= \langle \phi_\ell\ R_{\bar{\ell}sis}\ ,\phi_i\rangle$$

$$= \langle \phi_\ell\ R_{\bar{\ell}si}{}^{s}\ ,\phi_i\rangle$$

$$= \langle \phi_\ell\ R_{\bar{\ell}i},\quad \phi_i\rangle$$

Consequently $A \le \mathrm{Ricci}^M \le B$ implies that

$$A\langle \phi_i,\phi_i\rangle \le \langle d\phi(R^M(\bar{e}_s,e_i)e_s, d\phi(e_i)\rangle \le B\langle\phi_i,\phi_i\rangle$$

or otherwise said $Ae(\phi) \le \langle d\phi(R^M(\bar{e}_s,e_i)e_s, d\phi(e_i)\rangle \le Be(\phi)$.

This corollary is the basis of results like the following.

2.13. PROPOSITION [20]

Suppose $\phi : M \to N$ is a holomorphic mapping between Kähler manifolds with M compact. If $\mathrm{Ricci}^M \geq 0$ and is not identically zero and $\mathrm{HBRiem}^N \leq 0$ then $\phi$ is constant. If M has complex dimension 1 this last condition can be replaced by $\mathrm{HRiem}^N \leq 0$.

Proof

By the Corollary $e(\phi)$ is subharmonic and because M is compact the maximum principle for $\Delta$ forces $e(\phi)$ to be constant, and hence harmonic. Thus all terms in (4) being non-negative must vanish. However if $\mathrm{Ricci}^M > 0$ at some point the proof of 2.13 shows that at this point $e(\phi) = 0$. By its constancy $e(\phi) \equiv 0$ and so $\phi$ is constant.

(d) Complex dimension one

In this section M and N both have complex dimension one and $\phi : (M,g) \to (N,h)$ is harmonic. In this case formulae (4) and (5) can be written in another form. Let $K^M$, $K^N$ denote the indicated Gaussian curvature, then

$$\tfrac{1}{2}\Delta e'(\phi) = |\nabla d'\phi|^2 + K^M e'(\phi) - (\phi^* K^N) e'(\phi) J_\phi \tag{7}$$

$$\tfrac{1}{2}\Delta e''(\phi) = |\nabla d''\phi|^2 + K^M e''(\phi) - (\phi^* K^N) e''(\phi) J_\phi \tag{8}$$

where $J_\phi$ is the Jacobian of $\phi$ and is given by $J_\phi = e'(\phi) - e''(\phi)$. The following is also given in [27,29]

2.14. PROPOSITION

Away from the zeros of $e'(\phi)$ and $e''(\phi)$ respectively

$$\Delta \log e'(\phi) = 2K^M - 2(\phi^* K^N)J_\phi \tag{9}$$

$$\Delta \log e''(\phi) = 2K^M + 2(\phi^* K^N)J_\phi \ . \tag{10}$$

Proof

Choose local complex orthonormal frame fields on M and N and denote the components of the various covariant derivatives of $\phi$ with respect to these frames by $\phi_\theta = (\bar{\phi}_{\bar{\theta}})$ etc. Then

$$e'(\phi) = |\phi_\theta|^2 = \phi_\theta \bar{\phi}_{\bar{\theta}} \tag{11}$$

$$|\nabla d'\phi|^2 = |\phi_{\theta\theta}|^2 = \phi_{\theta\theta} \bar{\phi}_{\bar{\theta}\bar{\theta}} \tag{12}$$

since the harmonic equations read $\phi_{\theta\bar{\theta}} = \phi_{\bar{\theta}\theta}$ . (13)

Thus $\Delta \log e'(\phi) = 2(\log e'(\phi))_{\theta\bar{\theta}}$ by 2.9,

$$= \frac{\Delta e'(\phi)}{e'(\phi)} - \frac{2e'(\phi)_\theta e'(\phi)_{\bar{\theta}}}{(e'(\phi))^2}$$

$$= \frac{|\nabla d'\phi|^2}{e'(\phi)} - \frac{2|\phi_\theta|^2_\theta |\phi_\theta|^2_{\bar{\theta}}}{(e'(\phi))^2} + 2K^M - 2(\phi^* K^N)J_\phi$$

by (7) and (11)

$$= \left(\frac{|\nabla d'\phi|^2}{e'(\phi)} - \frac{2\phi_{\theta\theta}\bar{\phi}_{\bar{\theta}}\bar{\phi}_{\bar{\theta}\bar{\theta}}\phi_\theta}{(e'(\phi))^2}\right) + 2K^M - 2(\phi^* K^N)J_\phi$$

using (13)

$$= 2K^M - 2(\phi^* K^N)J_\phi \qquad \text{by (11), (12).}$$

The calculation of $\Delta \log e''(\phi)$ is completely parallel.

As mentioned before these formulae are given in [27, 29] where significant use is made of them in the case that N has non-positive curvature. There follows, here, an application of them to the case when N has constant positive curvature.

Consider a map $\phi : M \to S_1^2$, where $S_r^2 = \{x \in \mathbb{R}^3 ; |x| = r\}$ carries its induced metric of constant curvature $\frac{1}{r^2}$ and let $\Phi$ denote the composition $\Phi : M \xrightarrow{\phi} S_1^2 \hookrightarrow \mathbb{R}^3$. Then by 1.9 $\phi$ is harmonic if and only if $\Delta\Phi \perp S_1^2$, otherwise said this means that there exists a function $\lambda, M \to \mathbb{R}$ such that $\Delta\Phi = -\lambda\Phi$ on M.

2.15. DEFINITION

A map $\phi : M \to S_1^2$ is called a *eigenmap* if there exists a constant $\lambda$ such that $\Delta\Phi = -\lambda\Phi$, $\lambda$ is called the *eigenvalue* of $\phi$.

Note that an eigenmap is necessarily harmonic.

The following is quoted from [6].

2.16. THEOREM

Suppose that M is homeomorphic to $S^2$ and that $\phi_1, \phi_2, \phi_3$ are three first eigenfunctions such that their square sum is constant. Then M is actually isometric to a sphere with constant sectional curvature.

Cheng's proof proceeds by deriving an estimate for the first eigenvalue and then applying a result of J. Hersch concerning this estimate. However, using the method of harmonic maps all this can be dispensed with and in fact a better result can be proved along the lines that the eigenfunctions themselves generate the required isometry.

2.17. THEOREM

Suppose $(M,g)$ is a Riemannian 2-manifold of genus zero which admits an eigenmap $\phi : M \to S^2_1$, with eigenvalue $\lambda \neq 0$. Then the map $m \to \tilde{\Phi}(m) = \sqrt{2}/\lambda\ \Phi(m)$ furnishes an isometry between $(M,g)$ and the Euclidean sphere of radius $\sqrt{2}/\lambda$.

The proof of this result is divided into a series of lemmas.

Lemma A [37]

If $h$ denotes the metric on $S^2_1$ induced from its embedding in $\mathbb{R}^3$ then $\phi^*h = \rho g$ where $\rho : M \to \mathbb{R}$ is non-negative, in fact $\rho = |d\phi|^2$.

Proof

First note that $(M,g)$ admits a unique complex structure which is compatible both with $g$ and with some chosen orientation of $M$, this is the content of the existence of isothermal parameters. With respect to this complex structure decompose

$$\phi^*h = (\phi^*h)^{2,0} + (\phi^*h)^{1,1} + \overline{(\phi^*h)^{2,0}} .$$

Using the notation of 2.14 and denoting the dual coframe field on $M$ by $\omega$, $(\phi^*h)^{2,0} = \phi_\theta\, \bar{\phi}_\theta\, \omega^2$ . (14)

However $(\phi_\theta\, \bar{\phi}_\theta)_{\bar{\theta}} = \phi_{\theta\bar{\theta}}\, \bar{\phi}_\theta + \phi_\theta\, \bar{\phi}_{\theta\bar{\theta}} = 0$ by the harmonic equations (13) so $(\phi^*h)^{2,0}$ is a holomorphic quadratic differential on the Riemann sphere and hence is identically zero.

Thus $\phi^*h = (\phi^*h)^{1,1} = (|\phi_\theta|^2 + |\phi_{\bar{\theta}}|^2)\, \omega\, \bar{\omega} = (|\phi_\theta|^2 + |\phi_{\bar{\theta}}|^2) g$ which proves the lemma.

Lemma B

The function $\rho$ is a non-zero constant.

Proof

Since $|\Phi|^2 = 1$ on M the following equation holds

$$0 = \Delta|\Phi|^2 = 2\langle\Delta\Phi,\Phi\rangle + 2|d\phi|^2.$$

However, $\phi$ is an eigenmap so $\Delta\Phi = -\lambda\Phi$ and thus $|d\phi|^2 = \lambda$, together with Lemma A this proves Lemma B.

Now construct a complex structure on $S_1^2$ as for M.

Lemma C [18,37]

The map $\phi : M \to S_1^2$ is $\pm$ holomorphic.

Proof

From (11), (14) and the fact that $(\phi^*h)^{2,0} \equiv 0$ it is clear that at every point of M either $e'(\phi) = 0$ or $e''(\phi) = 0$. Let $Z'$ be the set on which $e'(\phi) = 0$ and $Z''$ that on which $e''(\phi) = 0$, and assume both are non-empty. Then

(a) $M = Z' \cup Z''$

(b) $Z'$ and $Z''$ are both closed

(c) $Z' \cap Z'' = \{m \in M;\ |d\phi|_m^2 = 0\}$.

By Lemma B $Z' \cap Z'' = \emptyset$ so $M = Z' \cup Z''$ provides a decomposition of M into closed sets contradicting the connectivity of M. Thus one of $Z'$ and $Z''$ is empty showing that $\phi$ is $\pm$ holomorphic.

By changing the orientation on M if necessary it may be assumed that $\phi$ is holomorphic.

Lemma D.

$\phi$ is a diffeomorphism.

Proof.

By using the fact that $e'(\phi) = e(\phi) = J(\phi) = \lambda/2$, formula (9) reads

$$0 = 2K^M - 2(\phi^*K^N)J_\phi .$$

Integrating this formula and applying the Gauss-Bonnet theorem yields the fact that the Brower degree of $\phi$ is 1. But $d\phi$ always has rank 2 to $\phi$ is a diffeomorphism.

To complete the proof, let $\tilde{\phi}$ be related to $\tilde{\Phi}$ as $\phi$ is to $\Phi$ and let $\tilde{h}$ denote the metric on the sphere of radius $\sqrt{2}/\lambda$ in $\mathbf{R}^3$ induced by that embedding. It is clear that $\tilde{\phi}$ is a diffeomorphism and that there exists a function $\mu$ such that $\tilde{\phi}^*\tilde{h} = \mu g$. There remains to determine the function. On one hand $e(\tilde{\phi}) = (2/\lambda)e(\phi) = 1$ and on the other $e(\tilde{\phi}) = \frac{1}{2}$ trace $\tilde{\phi}^*\tilde{h} = \frac{1}{2}$ trace $\mu g$, therefore $\mu \equiv 1$ and so $\tilde{\phi}$ is an isometry.

Remark

There is an alternative proof of Lemma D using facts about covering spaces as opposed to integral formulae.

Indeed, the fact $J(\phi) = \lambda/2$ shows that $\phi$ is a covering map and the simple connectivity of $S^2$ forces it to be a diffeomorphism.

In this way Theorem 2.17 and its Corollary 2.16 become transparent to a topologist.

# CHAPTER 3

## NON COMPACT DOMAINS

Putting aside the question of the existence of harmonic mappings from non-compact domains there remain two basic questions to be asked. Supposing such a mapping exists, can it have finite energy without reducing to a constant? Secondly, with reasonable conditions on the map can we say that its energy density is bounded and thus derive some geometric property of the map? This second question leads to a generalisation of the classical Schwarz lemma of complex variable theory.

### (a) Dilatation

As usual let $(M,g)$, $(N,h)$ denote two Riemannian manifolds and $\phi:M \to N$ any smooth mapping. Let $p \in M$ and consider $(\phi^*h)_p$. This is a symmetric non-negative bilinear form on $T_pM$ and can thus be put in the form $(\phi^*h)_p = \sum_{i=1}^{m} \lambda_i\, \omega_i \otimes \omega_i$, where $\omega_1,\ldots,\omega_m$ is an orthonormal basis of $T_p^*M$, and $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_m \geq 0$.

3.1. DEFINITION

The *dilatation* of $\phi$ at p is the quantity $(\lambda_1/\lambda_2)^{\frac{1}{2}}$ provided the rank of $\phi$ at p is at least 2. The map $\phi:M \to N$ has *dilatation bounded by* K if either $d\phi(p) = 0$ or $(\lambda_1/\lambda_2)^{\frac{1}{2}} \leq K$ at all points of M. Such a map has no point of rank 1.

The following will be of use later on.

3.2. LEMMA [39]

Suppose $\phi:(M,g) \to (N,h)$ has dilatation bounded by K and that its rank at each point of M is at most equal to k. If $\Lambda^2 d\phi$ denotes the induced map on 2-vectors then $|d\phi|^2 \leq kK\, |\Lambda^2 d\phi|$.

Proof

First note that $|d\phi|^2 = \sum_{i=1}^{m} \lambda_i$

and $|\Lambda^2 d\phi|^2 = \sum_{i<j} \lambda_i\lambda_j$ .

Thus $$\frac{\sum_{i=1}^{m} \lambda_i}{(\sum_{i<j} \lambda_i\lambda_j)^{\frac{1}{2}}} \le \frac{k\ \lambda_1}{(\sum_{i<j} \lambda_i\lambda_j)^{\frac{1}{2}}} \le \frac{k\ \lambda_1}{(\lambda_1\lambda_2)^{\frac{1}{2}}} \le k(\frac{\lambda_1}{\lambda_2})^{\frac{1}{2}} \le kK.$$

3.3. LEMMA [39]

If $(M,g)$, $(N,h)$ are Hermitian and $\phi:M \to N$ is holomorphic then it has dilatation bounded by 1.

Proof

Let $p \in M$, there is no loss in assuming $d\phi(p) \neq 0$. Suppose $(\phi^*h)_p = \sum_{i=1}^{2m} \lambda_i\ \omega_i \otimes \omega_i$ with $\lambda_1 \ge \lambda_2 \ge \dots \ge \lambda_{2m} \ge 0$. Let $e_i$ be the vector dual to $\omega_i$ so that $\{e_1,\dots,e_{2m}\}$ is an orthonormal basis of $T_pM$. The hypothesis that $\phi$ is holomorphic implies that

$$\phi^*h(Je_1,Je_1) = \phi^*h(e_1,e_1) = \lambda_1 \ . \qquad (1)$$

Now $Je_1$ is orthogonal to $e_1$ so it is a linear combination of $\{e_2,\dots,e_{2m}\}$, thus $Je_1 = \sum_{i=2}^{2m} \mu_i . e_i$ with $\sum_{i=2}^{2m} \mu_i^2 = 1$.

However $\phi^*h(Je_1,Je_1) = \sum_{i=2}^{2m} \mu_i^2 \; \phi^*h(e_i,e_i)$

$$= \sum_{i=2}^{2m} \mu_i^2 \lambda_i$$

$$\leq \sum_{i=2}^{2m} \mu_i^2 \lambda_2$$

$$= \lambda_2 \quad .$$

Coupled with (1) this gives $\lambda_1 = \phi^*h(Je_1,Je_2) \leq \lambda_2$, but by assumption $\lambda_2 \leq \lambda_1$ so the equality $\lambda_1 = \lambda_2$ holds as required.

(b) Bounds for the derivative

In this section the main result on harmonic maps of bounded dilatation is proved. The first proposition dispenses with the case of a compact domain.

3.4. PROPOSITION [40]

Let (M,g), (N,h) be Riemannian manifolds and suppose that there exist constants $A \geq 0$, $B > 0$ such that $\text{Ricci}^M \geq -A$ and $\text{Riem}^N \leq -B$. Suppose that $\phi:M \to N$ is a harmonic mapping of dilatation bounded by K. If $|d\phi|^2$ attains its maximum on M then it satisfies the inequality

$$|d\phi|^2 \leq \frac{k^2K^2}{2} \; \frac{A}{B}$$

where k is a bound for the rank of $\phi$ on M.

Proof

The proof is based on the following differential inequality of Eells-Sampson [ 8 ]

$$\tfrac{1}{2}\Delta|d\phi|^2 \geq \mathrm{Ricci}^M(d\phi,d\phi)-R^N(d\phi(v),d\phi(w),d\phi(v),d\phi(w)) \qquad (2)$$

whose repeated vectors indicate a trace is to be taken. By hypotheses and Lemma 3.3

$$\mathrm{Ricci}^M(d\phi,d\phi) \geq -A|d\phi|^2 \qquad (3)$$

$$-R^N(d\phi(v),d\phi(w),d\phi(v),d\phi(w)) \geq 2B|\Lambda^2 d\phi|^2 \geq \frac{2B}{k^2K^2}|d\phi|^4. \qquad (4)$$

Now let $p \in M$ be a point at which $|d\phi|^2$ attains its maximum, then $\Delta|d\phi|^2 \leq 0$ at p and this together with (2), (3), (4) gives

$$0 \geq -A|d\phi|^2 + \frac{2B}{k^2K^2}|d\phi|^4 \quad .$$

Rearranging this shows that at p $|d\phi|^2 \leq \frac{k^2K^2}{2}\,A/B$, but if this estimate holds at a maximum of $|d\phi|^2$ then it holds everywhere on M.

The above result has the major drawback of the need for the assumption that $|d\phi|^2$ actually attains its maximum and it is this which will be removed in due course, the idea is to use the following minimum principle of Omori-Yau.

3.5. MINIMUM PRINCIPLE [36]

Let (M,g) be a complete Riemannian manifold whose Ricci curvature is bounded below and let $f:M \to \mathbb{R}$ be a $C^2$ function which is bounded below. Then for any $\varepsilon > 0$ there exists a point $p \in M$ such that at p.

(i) $|df|^2 < \varepsilon$

(ii) $\Delta f > -\varepsilon$

(iii) $f < \inf . f + \varepsilon$.

What this principle says is that there exists a sequence of points of M along which the function f tends to its minimum and moreover its first and second derivatives are behaving as if these points were approaching a true minimum of f.

As indicated above this principle lies at the heart of the main result which follows. The idea of the proof is due to Yau [ 35 ], the result has been proved independently in [10].

3.6. THEOREM

Let $(M,g)$, $(N,h)$ be Riemannian manifolds with the following properties:

(a) M is complete and $\mathrm{Ricci}^M \geq -A$, some $A \geq 0$

(b) $\mathrm{Riem}^N \leq -B$ some $B > 0$.

If $\phi: M \to N$ is a harmonic map of dilatation bounded by K and rank bounded by k then

$$|d\phi|^2 \leq \frac{k^2K^2}{2} \, \frac{A}{B} \, .$$

In particular if $A = 0$ $\phi$ must be constant.

Proof

By (2), (3), (4) of Proposition 3.4.

$$\tfrac{1}{2}\Delta|d\phi|^2 \geq -A|d\phi|^2 + \frac{2B}{k^2K^2}|d\phi|^4. \tag{5}$$

Writing $P(|d\phi|^2)$ for the quantity on the right, the aim of the proof is to show that $P(|d\phi|^2) \le 0$.

Let $c > 0$ be an arbitrary but fixed constant, $|d\phi|^2 + c$ has a smooth square root and the function $f = (|d\phi|^2 + c)^{-\frac{1}{2}}$ is smooth and positive. At the centre of a system of normal coordinates

$$|df|^2 = \frac{\sum_{i=1}^{m} \left|\frac{\partial}{\partial x^i} |d\phi|^2\right|^2}{4(|d\phi|^2 + c)^3}$$

$$\Delta f = -\frac{\Delta|d\phi|^2}{2(|d\phi|^2+c)^{3/2}} + \frac{3}{4(|d\phi|^2+c)^{5/2}} \sum_{i=1}^{m} \left|\frac{\partial}{\partial x^i} |d\phi|^2\right|^2$$

so $$\Delta f = -\frac{\Delta|d\phi|^2}{2(|d\phi|^2+c)^{3/2}} + \frac{3|df|^2}{f}$$

and from (5)

$$\Delta f \le -\frac{P(|d\phi|^2)}{(|d\phi|^2+c)^{3/2}} + \frac{3|df|^2}{f} \qquad (6)$$

and this holds everywhere on M.

Since f is bounded below the maximum principle 3.5 can be applied. Let $\varepsilon > 0$ and find $p \in M$ so that at p

$|df|^2 < \varepsilon$

$\Delta f > -\varepsilon$

$f < \inf f + \varepsilon$.

Multiplying (6) by f and using these estimates yields

$$\frac{P(|d\phi|^2)}{(|d\phi|^2+c)^2} \leq \varepsilon(\inf f + \varepsilon) + 3\varepsilon \quad . \tag{7}$$

Now let $\varepsilon \to 0$ so that $f \to \inf f$ and $|d\phi|^2 \to \sup |d\phi|^2$. If $|d\phi|^2$ is unbounded then $\frac{P(|d\phi|^2)}{(|d\phi|^2+c)^2} \to \frac{2B}{k^2K^2}$ contradicting (7), hence $|d\phi|^2$ is bounded. A glance at (7) now reveals that $P(\sup|d\phi|^2) \leq 0$, and from the definition of P this means

$$-A \sup |d\phi|^2 + \frac{2B}{k^2K^2} \sup|d\phi|^4 \leq 0$$

giving $\sup|d\phi|^2 \leq \frac{k^2K^2}{2} \frac{A}{B}$ as required.

A corollary of this result will be needed in a later section.

3.7. COROLLARY

A complete Kähler manifold of non-negative Ricci curvature admits no non-constant bounded holomorphic functions.

Proof

Let M be such a manifold and $\Phi:M \to \mathbb{C}$ bounded and holomorphic. Without loss of generality it can be assumed that $|\phi| < 1$ so that $\phi$ maps M into the disc $D = \{z \in \mathbb{C};\ |z| < 1\}$ with its Poincaré metric of constant negative curvature. Since the holomorphic map $\phi:M \to D$ is harmonic with dilatation bounded by 1, the theorem implies it is constant.

3.8. COROLLARY [3]

Suppose (M,g) is a complete, simply connected Riemannian 2-manifold with non-negative curvature, then (M,g) admits no bounded harmonic functions.

Proof

Suppose $u: M \to \mathbb{R}$ is bounded and harmonic, its conjugate harmonic function $v$ is defined as follows. Let $x_o \in M$, then

$v(x) = \int_{[x_o,x]} *du$ where $[x_o,x]$ is any piecewise smooth path

from $x_o$ to $x$ and $*$ is the Hodge star operator, another choice of $x_o$ alters $v(x)$ by a constant. The function $w = u+iv$ is holomorphic and maps $M$ to a strip $-L < \mathrm{Re}\, w < L$ in the complex plane, call this strip $S$. By Riemann's mapping theorem there exists a holomorphic diffeomorphism $f: S \to D$ and so by 3.7 the composition $\begin{array}{ccc} M & \to S & \to D \\ m & \to w(m) & \to f(w(m)) \end{array}$ is constant thus $u$ is constant.

Having seen that Theorem 3.6 implies some vanishing theorems it is interesting to verify that it generalises the classical Schwarz lemma in the formulation given by Ahlfors.

3.9. COROLLARY

A holomorphic map $\phi: D \to D$ satisfies

$$\frac{|\phi'(z)|}{1-|\phi(z)|^2} \le \frac{1}{1-|z|^2}$$

where $|w|$ indicates the norm of the complex number $w$.

Proof

The Poincaré metric on D is given by the expression

$\dfrac{dx^2+dy^2}{(1-|z|^2)^2}$ with $z = x+iy$. An application of the Cauchy-Riemann

equations to $\phi$ yields

$$e(\phi) = \frac{(1-|z|^2)^2}{(1-|\phi(z)|^2)^2} \ |\phi'(z)|^2 \ . \tag{8}$$

However, Theorem 3.6 with $K = 1$, $k = 2$, $A = B$ shows that

$$e(\phi) = \tfrac{1}{2}|d\phi|^2 \leq 1,$$

which, together with (8) proves the result.

(c) The complex case

In this section the aspects of the theory which concern only those harmonic maps which are holomorphic are discussed. The final result is that of Yau [ 35 ] the original proof of which inspired the proof of Theorem 3.6.

3.10 THEOREM

Let $(M,g)$ be a complete Kähler manifold with $\mathrm{Ricci}^M \geq -A$, some $A \geq 0$, and $(N,g)$ another Kähler manifold with $\mathrm{HBRiem}^N \leq -B$, $B > 0$. If $\phi:M \to N$ is holomorphic then

$$|d\phi|^2 \leq 2A/B.$$

In particular if $A = 0$ $\phi$ is constant.

Proof.

The proof is similar to that of Theorem 3.6 with the exception that the starting point is the following inequality deduced from (6), Proposition 2.12,

$$\tfrac{1}{4}\Delta|d\phi|^2 \geq \langle d\phi(R^M(\bar{e}_s,e_i)e_s, d\phi(e_i)\rangle$$
$$- R^N(\overline{d\phi(e_s)}, d\phi(e_i), d\phi(e_s), \overline{d\phi(e_i)})$$

with the notation, as in Proposition 2.12, $\{e_1,\ldots,e_m\}$ is an orthonormal frame in $T^{1,0}M$. The analogues to (3) and (4) of this chapter are

$$\langle d\phi(R^M(\bar{e}_s,e_i)e_s, d\phi(e_i)\rangle \geq -\frac{A}{2}|d\phi|^2$$

$$-R^N(\overline{d\phi(e_s)}, d\phi(e_i), d\phi(e_s), \overline{d\phi(e_i)}) \geq \frac{B}{4}|d\phi|^4$$

which have been deduced in the proof of Proposition 2.12. Consequently

$$\Delta|d\phi|^2 \geq -2A|d\phi|^2 + B|d\phi|^4 \quad \text{replaces (5)}$$

and the proof continues as before to obtain $|d\phi|^2 \leq \frac{2A}{B}$.

Remark

If M has complex dimension 1 the hypothesis $\mathrm{HBRiem}^N \leq -B$ can be replaced by $\mathrm{HRiem}^N \leq -B$ exactly as in Chapter 2.

(d) Is Energy finite?

Returning now to the questions posed at the beginning of this chapter, let us consider the question of what sort of conditions on the manifolds and a harmonic map $\phi$ between them force this map to have infinite energy. One theorem in this vein is due to Schoen-Yau [28].

3.11. THEOREM

Suppose $(M,g)$ is a complete Riemannian manifold with $\mathrm{Ricci}^M \geq 0$ and $(N,h)$ a Riemannian manifold with $\mathrm{Riem}^N \leq 0$. If $\phi:(M,g) \to (N,h)$ is harmonic with $E(\phi) < \infty$ then $\phi$ is constant.

As suggested above this theorem is essentially proved in [28]. Another theorem in this direction is given in [ 9 ]. Let $S^n$ denote the unit n-sphere in $\mathbb{R}^{n+1}$.

3.12. THEOREM

Suppose $\phi:\mathbb{R}^m \to S^n$ is harmonic and $m \geq 3$. If $E(\phi) < \infty$ then $\phi$ is constant.

This will be proved later on as part of a more general theorem.

Noting that $\mathbb{R}^m$ has non-negative Ricci curvature it is tempting to ask whether the condition $\mathrm{Riem}^N \leq 0$ in Theorem 3.11 is really necessary, the answer is affirmative according to the following example.

3.13. Example

The aim of this example is to show the existence of a complete, non-compact manifold M with $\mathrm{Ricci}^M \geq 0$ and a finite energy harmonic map $\phi:M \to S^2$ which is not a constant map. Note that $S^2 = N$ violates the condition $\mathrm{Riem}^N \leq 0$.

The starting point of the example is a parabaloid of revolution P.

For example P would be the set of points
$\{(ax,ay,a^2(x^2+y^2)) \in \mathbb{R}^3;\ (x,y) \in \mathbb{R}^2,\ a > 0\}$. As has been observed in the previous chapter the surface P has an induced conformal structure.

Lemma A.

P admits a holomorphic diffeomorphism $u:P \to \mathbb{C}$.

Proof

Either by a direct calculation or by inspecting the picture it is easy to see that P has positive curvature and is complete so by 3.7 it admits no bounded holomorphic functions. On the other hand by the uniformization theorem for simply connected Riemann surfaces, P admits a holomorphic diffeomorphism onto either the unit disc or the whole complex plane. The reasoning above has precluded the first possibility.

The next step is to find a non-constant, finite energy harmonic map $\psi:\mathbb{C} \to \mathbb{C}$ and to use u of Lemma A to pull it back to P as a map with the same properties.

Lemma B. [8]

Let $(M,g)$, $(M',g')$ and $(N,h)$ be Riemannian manifolds with dimension M = dimension M' = 2. If $V \subset\subset M'$ and $u:V \to M$ is a holomorphic diffeomorphism onto its image then

$$E(\psi \circ u, V) = E(\psi, u(V))$$

for any smooth mapping $\psi:M \to N$.

Proof

As u is holomorphic 3.2 shows that $u^*g = \rho g'$ for some positive function $\rho$, since u is a diffeomorphism it is strictly positive. A calculation shows that

$$e(\psi \circ u) = \tfrac{1}{2}\langle d(\psi \circ u), d(\psi \circ u)\rangle = \rho e(\psi) \circ u = J_u(e(\psi) \circ u)$$

and so

$$E(\psi \circ u), V) = \int_V e(\psi \circ u) dv_g = \int_V e(\psi) \circ u . J_u dv_g$$

$$= \int_V u^*(e(\psi) dv_g) = \int_{u(V)} e(\psi) dv_g = E(\psi, u(V)).$$

As a consequence of this lemma if $\psi:\mathbb{C} \to S^2$ is harmonic then $\psi \circ u : P \to \mathbb{C}$ is also harmonic and $E(\psi) = E(\psi \circ u)$.

Lemma C.

The harmonic maps $\psi:S^2 \to S^2$ are all given in complex coordinates z by rational expressions in z or $\bar{z}$.

This has essentially been established in Lemma A of Theorem 2.17, the complete details may be found in [ 18 ].

To continue with the example, let $\psi:S^2 \to S^2$ be one of the non-constant holomorphic harmonic maps provided by Lemma C and writing $S^2 : \mathbb{C} \cup \{\infty\}$ let $\psi$ also denote the restriction $\psi:\mathbb{C} \to S^2$. Then by the remark following Lemma B the composition $\tilde{\phi}:P \xrightarrow{u} \mathbb{C} \xrightarrow{\psi} S^2$ is harmonic with $E(\tilde{\phi}) = E(\psi) < \infty$.

Now let (X,k) be any complete Riemannian manifold with the properties $\mathrm{Ricci}^X \geq 0$, $\mathrm{Vol}(X) = \int_X dv_k < \infty$ (for example any irreducible symmetric space of compact type will do [16,II]), and define

$$\phi : P\times X \to S^2$$

by $\phi(p,x) = \tilde{\phi}(p)$, then

(i) $\tau(\phi) = \tau^P(\phi) + \tau^X(\phi)$ by 1.7

$= 0 + 0$ by construction.

Thus $\phi$ is harmonic with respect to the product metric.

(ii) $E(\phi) = \int_{P\times X} e(\phi)(p,x)dv_g \wedge dv_h = \int_{P\times X} e(\tilde{\phi})(p)dv_g \wedge dv_h$ since $\phi$ is constant in X

$= \int_X (\int_P e(\tilde{\phi})dv_g)dv_h$ by Fubini's theorem

$= \int_X E(\psi)dv_h = \mathrm{Vol}(X)E(\psi) < \infty.$

Thus $\phi$ has finite energy.

(iii) If $M = P\times X, \mathrm{Ricci}^M > 0$, the standard calculation is omitted.

(iv) M is complete.

According to (i) - (iv) $\phi : M \to S^2$ realises the aim of the example.

Remark

Note that as $\phi$ is holomorphic in one variable and constant in the other it has dilatation bounded by 1 so the situation is in no way altered by requiring the map to have bounded dilatation.

This example brings to light several points. Firstly, it shows that Theorems 3.11 and 3.12 are independent of each other, i.e. there is no wider theorem embracing them both. Secondly, the construction of the example itself indicates the key to the hypothesis in Theorem 3.12 that dim $M \geq 3$, which is the conformal invariance of the energy in 2 dimensions as in Lemma B. A result similar to 3.12 and also taking advantage of the non-invariance of the energy under conformal transformation in dimensions higher than 2 will be introduced, but first some defintions are needed.

Suppose that M is an open disc neighbourhood of O in $\mathbb{R}^m$ and that $g = f^2(x)\sum_{i=1}^{m}(dx^i)^2$ is a Riemannian metric on M which is a conformal deformation of the standard flat metric. Denote by r the Euclidean distance function centred at O, i.e. $r(x) = (\sum_{i=1}^{m}(x^i)^2)^{\frac{1}{2}}$. $S_r$ is the sphere of radius r and centre O and $\alpha_r$ will denote its second fundamental form in (M,g) as in [ 16 , II p.10-12]. Define a vector field H on M by

$$H_x = \frac{1}{m-1}\,\mathrm{Trace}(\alpha_r)_x = \text{mean curvature normal of } S_r \text{ at } x$$

for $x \in S_r$.

3.14. <u>PROPOSITION</u>

$$H_x = -\frac{1}{r}\,(1 + r\partial/\partial r \log f)\, f^{-2}(x)\,\partial/\partial r.$$

<u>Proof</u>

Considering $S_r$ as a level hypersurface of $r^2 : M \to \mathbb{R}$, with $N = \dfrac{\nabla r^2}{|\nabla r^2|^2}$ it is easy to see that

$$\alpha_r = -\nabla^2 r^2 . N. \tag{9}$$

Let $x \in S_r$ be the point $(r,0,\dots,0)$, then $\nabla^2 r^2$ is calculated at x. Let $\Gamma^k_{ij}$ be the components of the Levi-Civita connection associated to g, so $\Gamma^k_{ij} = f^{-1}(f_i\,\delta_{jk} + f_j\delta_{ik} - f_k\delta_{ij})$ the subscripts denoting differentation. By definition

$$\nabla^2 r^2(\partial/\partial x^i, \partial/\partial x^j) = \frac{\partial^2 r^2}{\partial x^i \partial x^j} - \frac{\partial r^2}{\partial x^k} \Gamma^k_{ij}$$

and evaluating at x

$$\nabla^2 r^2(\partial/\partial x^i, \partial/\partial x^i) = 2(1 + r \frac{\partial}{\partial r} \log f(x)) \text{ for } i = 2,\dots,m \quad (10)$$

A simple calculation shows that $N_x = \frac{1}{2r} \partial/\partial r$ and that $\{f^{-1}(x) \partial/\partial x^i\}_{i=2}^m$ is an orthonormal basis of $T_x S_r$, so that

$$H_x = \frac{1}{m-1} \operatorname{Trace}(\alpha_r)_x = \frac{f^{-2}(x)}{m-1} \sum_{i=2}^m (\alpha_r)_x(\partial/\partial x^i, \partial/\partial x^i)$$

$$= \frac{-f^{-2}(x)}{m-1} \sum_{i=2}^m (1 + r\, \partial/\partial r \log f(x))\, \partial/\partial r \quad \text{by (9), (10)}$$

$$= -\frac{1}{r} (1 + r \frac{\partial}{\partial r} \log f(x)) f^{-2}(x)\, \partial/\partial r$$

and the result holds at $(r,0,\dots,0)$. To obtain the result at arbitrary $x \in S_r$ one need only remark on the symmetry of the divided expression for $H_x$ with respect to the transitive action of $O(m)$ on $S_r$.

3.15. <u>COROLLARY</u>

$1 + r\, \partial/\partial r \log f \geq 0$ on M if and only if H is never pointing away from 0. Moreover H is not identically zero.

<u>Proof</u>.

The first statement follows directly from the proposition. For the second, suppose to the contrary that $1 + r\, \partial/\partial r \log f = 0$ on M, then $\partial/\partial r \log rf(x) = 0$, integration yields $f(x) = \frac{c}{r}$, where C depends only on $\frac{x}{|x|}$, which is absurd.

Now let $\sigma_r$ denote the volume form of the induced Riemannian metric on $S_r$ and set $V(r) = \int_{S_r} \sigma_r = \mathrm{Vol}(S_r)$.

Another characterisation of $1 + r\,\partial/\partial r \log f \geq 0$ is given .

3.16. PROPOSITION

If $1 + r\frac{\partial}{\partial r} \log f \geq 0$ on M then V is a non-decreasing function of r.

Proof

Let $\omega_r$ denote the Euclidean volume element on $S_r$ so that $\sigma_r = f^{m-1}\omega_r$, and $V(r) = \int_{S_r} f^{m-1}\omega_r$. A change of variables gives

$$V(r) = \int_{S_1} f^{m-1}(rx)r^{m-1}\omega_1 .$$

$$\text{Thus } V'(r) = (m-1)\int_{S_1} \left(f^{m-2}(rx)x^i \frac{\partial f}{\partial x^i}(rx)r^{m-1} + f^{m-1}(rx)r^{m-2}\right)\omega_1$$

$$= (m-1)\int_{S_r} r^{-1}(1 + r\,\partial/\partial r \log f)\,\omega_r$$

and the result is proved since $m - 1 \geq 0$.

Remark

This proposition could equally well have been deduced from the first variation formula for area given in [31 ] and the fact that $g(H,-r\,\partial/\partial r) = 1 + r\,\partial/\partial r \log f$ everywhere on M.

3.17. DEFINITION

Letting $\mathbb{R}^n$ denote real m-dimensional space and $D^m$ the open unit disc entered at O a Riemannian metric g on $\mathbb{R}^m$ will be said to be of type A if

(A) $g_x = f^2(x) \sum_{i=1}^{m} (dx^i)^2$ and H is never pointing away from O.

A Riemannian metric $\tilde{g}$ on $D^m$ will be said to be of type B if

(B) $\tilde{g}_x = \tilde{f}^2(r) \sum_{i=1}^{m} (dx^i)^2$, i.e. f(x) depends only on the distance of x from O , $\int_0^1 \tilde{f}^{m-2}(r)dr = \infty$ and $\tilde{H}$ is never pointing away from O, $\tilde{H}$ is the vector field corresponding to $\tilde{g}$ as H corresponds to g.

Finally, if $g_x = f^2(x) \sum_{i=1}^{m} (dx^i)^2$ is another metric on $D^m$, say that g dominates a metric of type B if there exists a metric $\tilde{g}$, as above, of type B such that at each point of $D^m$ $g(H,-r\ \partial/\partial r) \geq g(\tilde{H},-r\ \partial/\partial r)$. By the remark following 3.16 this is equivalent to $1 + r\ \partial/\partial \log f \geq 1 + r\ \partial/\partial r \log \tilde{f}$ at every point of $D^m$.

3.18. Examples

(1) The flat Euclidean metric is an example of a metric of type A on $\mathbb{R}^m$.

(2) The metric $\sum_{i=1}^{m} \frac{(dx^i)^2}{(1-r^2)^2}$ is an example of a metric type B on $D^m$ whenever $m \geq 3$. It is a model for a metric of constant negative sectional curvature, in fact its sectional curvature are all -4.

Notice that both these metrics are complete.

The main theorem of this section is the following. A form $\omega \in A^1(W)$ is of class $L^2$ if $\int_M |\omega|^2 \, dr_g < \infty$. Recall that if $\phi: M \to N$ is harmonic then $d\phi \in A'(\phi^*TN)$ satisfies $d(d\phi) = 0$ $d^*(d\phi) = 0$ and $\phi$ has $E(\phi) < \infty$ if and only if $d\phi$ is of class $L^2$.

3.19. THEOREM

Suppose that $(M,g)$ is a Riemannian manifold of one of the following types

(a) $M = \mathbb{R}^m$, $m \geq 3$ and g is of type A

(b) $M = D^m$ and g dominates a metric of type B.

If W is a Riemannian vector bundle over M then any 1-form $\omega \in A'(W)$ which is of class $L^2$ and satisfies $d\omega = 0$, $d^*\omega = 0$ must vanish identically.

Remarks

The metric in question is not necessarily complete. Indeed, in the case of a metric of type B on $D^m$ the exponential map at O is defined on the whole of $T_oD^m$ if and only if $\int_o^1 \tilde{f}(r)dr = \infty$.

Secondly note the lack of curvature hypotheses on W.

Thirdly, note that the hypotheses imply that $m \neq 2$. The proof of 3.19 will be given after a contrasting example. It has been hinted, and is indeed the case, that the heart of the theorem is the fact that in dimensions unequal to 2 the energy integral and $L^2$ norm of a 1-form are changed under conformal deformation of the domain. Thus the following may come as a surprise.

3.20. PROPOSITION

Suppose that $\mathbb{R}^2 = \mathbb{C}$ carries its standard flat metric and that $\phi:\mathbb{R}^2 \to \mathbb{R}^2$ is harmonic. If $E(\phi) < \infty$ then $\phi$ is constant.

Proof

Step 1, $\phi$ is $\pm$ holomorphic

Set h = flat metric on $\mathbb{R}^2$. Then using the indicated complex structure

$$\phi^*h = (\phi^*h)^{2,0} + (\phi^*h)^{1,1} + \overline{(\phi^*h)^{2,0}} \quad .$$

As in Lemma A of Theorem 2.17 $(\phi^*h)^{2,0}$ is a holomorphic quadradic differential on $\mathbb{R}^2$ and in fact

$$\phi^*h = \phi_z \, \bar{\phi}_z \, dz^2 .$$

To show that this vanishes the following inequality is used

$$\int_{\mathbb{R}^2} |\phi_z \, \bar{\phi}_z| \, dxdy \le \int_{\mathbb{R}^2} (|\phi_z|^2 + |\phi_{\bar{z}}|^2) dxdy = \int_{\mathbb{R}^2} e(\phi) dxdy = E(\phi) < \infty \; .$$

Thus the holomorphic function $\phi_z \bar{\phi}_z$ is of class $L^2$ on the whole plane so it vanishes by Liouville's theorem. Consequently

$$\phi^*h = (\phi^*h)^{1,1} = \rho^2 h .$$

But it is classical that the harmonic map $\phi$ has isolated critical points so the argument of Lemma C of 2.17 shows that $\phi$ is $\pm$ holomorphic.

Step 2, $\phi$ is constant

Without loss of generality assume that $\phi$ is holomorphic and so $\phi_{\bar{z}} \equiv 0$ on $\mathbb{R}^2$. Then, $\phi_z$ being the derivative of a holomorphic function, is itself holomorphic, so

$$\int_{\mathbb{R}^2} |\phi_z|^2 \, dxdy = \int_{\mathbb{R}} e(\phi)dxdy = E(\phi) < \infty$$

and $\phi_z$ vanishes. Thus $d\phi \equiv 0$ and $\phi$ is constant.

The fact is that this result holds because of the nature of the metric on the range space, it has nothing to do with that on the domain. Indeed in Example 3.13 a non-constant harmonic map

$$\phi:\mathbb{R}^2, \text{ flat} \to \mathbb{R}^2, h$$

where $h = \frac{dx^2+dy^2}{(1+r^2)^2}$ has constant positive curvature has been constructed.

Proof of 3.19.

The proof of this theorem follows in the form of a calculation the most important idea of which is the calculation of the first variation of a certain integral by two different methods. On one hand Green's theorem is used while on the other a direct calculation is made. For the second method a preparatory lemma is needed.

3.21. LEMMA

Suppose $G:\mathbb{R}^m \times \mathbb{R} \to \mathbb{R}$ is smooth. If $D_r$, $S_r$ denote the disc and sphere of radius r in $\mathbb{R}^m$ and $\omega_r$ denotes the Euclidean volume element on $S_r$, then

$$\frac{d}{dt} \int_{D_{tR}} G(x,t)dx \bigg|_{t=1} = R \int_{S_R} G(x,1)\omega_R + \int_{D_R} \frac{\partial G}{\partial t} (x,t)dx \bigg|_{t=1} \tag{11}$$

Proof

First set $f(r,t) = \int_{S_r} G(x,t)\omega_r$, $f(0,0) = 0$.

Then $\displaystyle \frac{d}{dt} \int_0^t f(r,t)dr \bigg|_{t=1} = \frac{d}{dt} \int_0^1 f(ts,t)tds \bigg|_{t=1}$

$$= \int_0^1 s \frac{\partial f}{\partial s} (s,1)dr + \int_0^1 \frac{\partial f}{\partial t}(s,t)ds \bigg|_{t=1} + \int_0^1 f(s,1)ds$$

$$= \int_0^1 \frac{d}{ds} (sf(s,1))ds + \int_0^1 \frac{\partial f}{\partial t} (s,t)ds \bigg|_{t=1}$$

$$= f(1,1) + \int_0^1 \frac{\partial f}{\partial t} (r,t)dr \bigg|_{t=1} . \tag{12}$$

By changing variables

$$\frac{d}{dt} \int_{D_{tR}} G(x,t)dx \bigg|_{t=1} = \frac{d}{dt} \int_0^t \left(\int_{S_r} G(Ry,t)R^m\omega_r\right)dr \bigg|_{t=1}$$

$$= R^m \int_{S_1} G(Ry,1)\omega_1 + \int_0^1 R^m\left(\int \frac{\partial G}{\partial t} (Ry,t)\omega_r\right)dr \bigg|_{t=1} \quad \text{by (12)}$$

$$= R \int_{S_R} G(x,1)\omega_R + \int_{D_R} \frac{\partial}{\partial t} G(x,t)dx \bigg|_{t=1} .$$

It will be useful to know later on that a non-zero harmonic 1-form cannot vanish on an open set. This follows from the unique continuation theorem of Aronszajn [ 3 ] which may be stated as follows. Let A be a linear elliptic second order differential operator defined on a domain U of $\mathbb{R}^n$. Let $u = (u^1, \dots, u^r)$ be functions satisfying the inequalities

$$|Au^\alpha| \le K \{ \sum_{\beta=1}^{r} |du^\beta| + \sum_{\beta=1}^{r} |u^\beta| \}. \tag{13}$$

If $u = 0$ in a subdomain of U then $u \equiv 0$ in U.

3.22 PROPOSITION.

Let W be a Riemannian vector bundle over M, and let $\omega \in A^1(W)$ satisfy $\Delta\omega = 0$. If $\omega$ vanishes on an open set then it vanishes on M.

Proof

Let U be some open set of M over which W is trivialised by the sections $s_1, \dots, s_r$. Define the functions $u^\alpha$, $1 \le \alpha \le r$ by $\omega = u^\alpha s_\alpha$ on U. Since $\omega$ is harmonic the real form of Theorem 2.10, given in [ 21 ], shows that $\mathrm{Tr}\nabla^2\omega = S(\omega)$ where $S(\omega)$ is linear in $\omega$ and depends only on the geometry of M and W. Then using the formula $\mathrm{Tr}\nabla^2 u^\alpha s_\alpha = (\Delta u^\alpha)s_\alpha + u^\alpha \mathrm{Tr}\nabla^2 s_\alpha + \mathrm{Tr}(du^\alpha . \nabla s_\alpha)$ it can be seen that

$$\Delta u^\alpha = \alpha^{\text{th}} \text{ component of } \{- \sum_\alpha u^\alpha(S(s_\alpha) + \mathrm{Tr}\nabla^2 s_\alpha) + \mathrm{Tr}(du^\alpha . \nabla s_\alpha)\}$$

in particular the $u^\alpha$ satisfy (13) on any relatively compact set.

Since $\omega$ vanishes on an open set so do the $u^\alpha$ and hence they and $\omega$, vanish on a slightly larger set. By the connectedness of M $\omega$ vanishes on M.

Let $\omega$ be as in the theorem with $\omega \not\equiv 0$, the aim is to show that $\int_M |\omega|^2 \, dr_g = \infty$.

To begin with $\omega$ is embedded in a one parameter family of W-valued 1-forms as follows. Choose $R > 0$ so that $S_R \subset M$ and for $0 < t < 1/R$ set

$$\omega_t(X_x) = t\ \tau_t^{-1}\omega(x_{tx}) \qquad\qquad |x| < R$$

where X is understood to be any constant vector field and $\tau_t^{-1}$ is parallel transport in W along the straight line from tx to x.

3.23. <u>LEMMA</u>

$$\frac{d}{dt}\,\omega_t\Big|_{t=1} = \nabla\omega(r\ \partial/\partial r). \tag{14}$$

<u>Proof</u>

It is well known that $\frac{d}{dt}\,\tau_t^{-1}\ \omega(X_{tx})\Big|_{t=1} = \nabla^W_{r\partial/\partial r}\ \omega(X\ )$

so that

$$\frac{d}{dt}\,\omega_t\Big|_{t=1}(X) = \omega(X) + \nabla^W_{r\partial/\partial r}\omega(X)$$

for all vector fields X. Now consider the vector field $X_i = \partial/\partial x^i$. Since $d\omega = 0$

$$0 = \nabla^W_{r\partial/\partial r}\ \omega(X_i) - \nabla^W_{X_i}\omega(r\partial/\partial r) - \omega([r\partial/\partial r, X_i]$$

while a simple calculation shows that $[r\ \partial/\partial r,\ X_i] = -\ X_i$.

Hence $\frac{d}{dt}\,\omega_t\Big|_{t=1}(X_i) = \omega(X_i) + \nabla^{W}_{r\partial/\partial r}\,\omega(X_i) = \nabla^{W}_{X_i}\,\omega(r\partial/\partial r)$ and the lemma is proved.

Now $\{f^{-1}(x)\,\partial/\partial x^i\}_{i=1}^{m}$ is an orthonormal basis of $T_xM$ so that

$$|\omega_t|^2_x \quad \langle\omega_t,\omega_t\rangle_x = \sum_{i=1}^{m} f^{-2}(x)\langle\omega_t(\partial/\partial x^i),\omega_t(\partial/\partial x^i)\rangle_x$$

$$= \sum_{i=1}^{m} t^2 f^{-2}(x)\langle\omega(\partial/\partial x^i),\omega(\partial/\partial x^i)\rangle_{tx}$$

where account has been taken of the fact that the parallel transport in question is isometric for the fibre metric of $W$. Thus

$$|w_t|^2_x = \frac{t^2 f^{-2}(x)}{f^{-2}(tx)} \quad |\omega|^2_{tx} \quad . \tag{15}$$

Letting $\sigma_r$ denote the volume element of the induced Riemannian metric on $S_r$ the equations $\sigma_r = f^{m-1}\omega_r$ and $dv_g = f dr \wedge \sigma_r$ hold on $S_r$. Define

$$A(r,t) = \int_{S_r} |\omega_t|^2\, f.\sigma_r \qquad \text{for } r \le R \tag{16}$$

and

$$\|\omega_t\|^2_R = \int_0^R A(r,t)dr = \int_{D_R} |\omega_t|^2\, dv_g. \tag{17}$$

Now $$\frac{d}{dt}\,\|\omega_t\|^2_R\Big|_{t=1} = 2\int_{D_R} \langle\omega, \frac{d\omega_t}{dt}\rangle\, dv_g\Big|_{t=1}$$

$$= 2\int_{D_R} \langle\omega,\nabla^{W}_{\omega}(r\partial/\partial r)\rangle\, dv_g \text{ by (14)}$$

$$= 2\int_{D_R} \langle d^*\omega,\omega(r\partial/\partial r)\rangle\, dv_g + 2\int_{S_R}\langle\omega(\partial/\partial r),\omega(r\partial/\partial r)\rangle\sigma_R$$

this last equality following by Green's theorem.

In particular, since $d{*}\omega = 0$ and $\omega(r\partial/\partial r) = r\omega(\partial/\partial r)$ it follows that

$$\frac{d}{dt} \, \|\omega_t\|_R^2 \Big|_{t=1} \geq 0 \quad . \tag{18}$$

On the other hand, substituting (15) into (17) and reparametrising the integral gives

$$\|\omega_t\|_R^2 = t^{2-m} \int_{D_{tR}} |\omega|_x^2 \, f^{m-2}(x/t) f^2(x) dx.$$

To differentiate this, set $G(x,t) = |\omega|_x^2 \, f^{m-2}(x/t) f^2(x)$

so that $\frac{\partial G}{\partial t}(x,t)\Big|_{t=1} = (2-m)\ |\omega|_x^2 (r\partial/\partial r \log f(x)) f^m(x).$

Applying (11) one obtains

$$\begin{aligned}\frac{d}{dt} \, \|\omega_t\|_R^2\Big|_{t=1} &= R\int_{S_R} |\omega|_x^2 f^m(x)\ \omega_R + (2-m)\int_{D_R} |\omega|_x^2 (1+r\frac{\partial}{\partial r}\log f(x)) dv_g \\ &= RA(R,1) + (2-m)\int_{D_R} |\omega|_x^2 (1+r\partial/\partial r \log f(x)) dv_g\end{aligned}$$

using the definition of $A(R,t)$. Applying (18) and rewriting the integral, this becomes

$$RA(R,1) \geq (m-2)\int_0^R \Big(\int_{S_r} |\omega|_x^2 \ (1+r\partial/\partial r \log f(x)) f(x)\sigma_r\Big) dr \tag{19}$$

and this holds for all $R$ in case $M=\mathbb{R}^m$ and all $R < 1$ in case $M = D^m$. To complete the proof the two cases are considered separately.

(a) In this case $M = \mathbb{R}^m$, $m \geq 3$ and $g$ is of type **A**.

Since it may be assumed that $f(x) > 0$ Corollary 3.15 and Proposition 3.22 together with (19) imply the existence of $c > 0$ and $R_o > 0$ such that for $R \geq R_o$ the estimate $RA(R,1) \geq c$ holds. Then

$$\int_{\mathbb{R}^m} |\omega|^2 \, dv_g = \int_o^\infty A(R,1)dR \geq \int_{R_o}^\infty A(R,1)dR \geq c\int_{R_o}^\infty \frac{dR}{R} = \infty$$

and the proof is complete in this case.

(b) In this case $M = D^m$ and $g$ dominates a metric $\tilde{g}$ of type B. Write that metric, $\tilde{g} = \tilde{f}^2(r) \sum_{i=1}^m (dx^i)^2$ and note that by the conditions imposed on $\tilde{f}$, $m \neq 2$. Suppose that $m \geq 3$, it follows from (19) that for all $0 < R < 1$

$$RA(R,1) \geq (m-2) \int_o^R \left(\int_{S_r} |\omega_x|^2 \, (1 + r\partial/\partial r \log \tilde{f}(x))f(x)\sigma_r\right)dr$$

$$\geq (m-2) \int_o^R A(r,1)(1 + r\partial/\partial r \log \tilde{f}(r))dr. \tag{20}$$

Setting $J(r) = (m-2)(1 + r\partial/\partial r \log \tilde{f}(r))$, $J(r) \geq 0$ for all $0 < r < 1$ and by Corollary 3.15 is not identically zero. Again by Proposition 3.22 there exists $0 < R_o < 1$ such that

$$(m-1)\int_o^{R_o} A(r,1)J(r)dr > 0.$$

Call this number $K_o$. The inequality (20) now reads

$$RA(R,1) \geq K_o + \int_{R_o}^{R} A(r,1)J(r)dr \quad \text{all } R_o < R < 1. \tag{21}$$

It will be shown that this implies

$$RA(R,1) \geq K_1 \tilde{f}^{m-2}(R) \tag{22}$$

for some $K_1 > 0$ and all $R_o < R < 1$, which inequality implies the result because

$$\int_{D^m} |\omega|^2 dv_g = \int_o^1 A(R,1)dR \geq \int_{R_o}^1 A(R,1)dR \geq \int_{R_o}^1 RA(R,1)dR$$

$$\geq K_1 \int_{R_o}^1 \tilde{f}^{m-2}(R)dR = \infty \quad \text{by hypothesis.}$$

There remains to show that (22) follows from (21). To this end set

$$Y(R) = K_o + \int_{R_o}^{R} A(r,1)J(r)dr \quad \text{for } R_o < R < 1$$

so that, using (21)

$$Y'(R) = A(R,1)J(R) \geq \frac{J(R)}{R} Y(R).$$

This implies that

$$\frac{d}{dR}\left[Y(R)\exp\left(-\int_{R_o}^{R} \frac{J(r)}{r}dr\right)\right] \geq 0$$

which inequality integrated over $(R_o,R)$ yields

$$Y(R) \geq K_o \exp\left(\int_{R_o}^{R} \frac{J(r)}{r}\, dr\right). \tag{23}$$

But $\exp\left(\int_{R_o}^{R} \frac{J(r)}{r}\, dr\right) = \exp\left[(m-2)\log \frac{R\tilde{f}(R)}{R_o\tilde{f}(R_o)}\right] = \frac{R^{m-2}\tilde{f}^{m-2}(R)}{R_o^{m-2}\ \tilde{f}^{m-2}(R_o)}$ ,

so setting $K_1 = K_o\ \tilde{f}^{2-m}(R_o)$ and using $R/R_o > 1$ (23) shows that

$$Y(R) \geq K_1\tilde{f}^{m-2}(R) \text{ for } R_o < R < 1.$$

Applying (21) and using the definition of $Y(R)$ gives

$$RA(R,1) \geq K_1\tilde{f}^{m-2}(R)$$

which is (22) and the result is proved.

# CHAPTER 4

## HARMONIC MAPS AND CURVATURE

This chapter is, in the first instance, devoted to several applications of the Eells-Sampson formula already used in Chapter 3. Namely, for a harmonic map $\phi:M \to N$ the following identity is valid

$$\Delta e(\phi) = |\nabla d\phi|^2 + \langle d\phi(\mathrm{Ricci}^M v), d\phi(v)\rangle - R^N(d\phi(v), d\phi(w), d\phi(v), d\phi(w)) \quad (1)$$

where a repeated vector indicates that a trace is to be taken.

### (a) Maps of small energy

In this section a study is made of a harmonic map $\phi:M \to N$ whose energy density is suitably bounded by the curvatures of M and N. The particular form of the estimate considered forces the map to be of a very special form topologically.

More specifically, the object of study is a harmonic map $\phi:(M,g) \to (N,h)$ for which the following hypotheses are valid.

(a) There exist numbers A, B > 0 such that $\mathrm{Ricci}^M \geq A$ and $\mathrm{Riem}^N \leq B$.

(b) M is complete, and hence compact by a theorem of Myers cf. [ 16, II,p.88]

(c) The map $\phi:M \to N$ satisfies $e(\phi) \leq A/B$ and rank $\phi \leq 2$.

The result to be proved is the following.

4.1. THEOREM

Suppose that $\phi:M \to N$ is harmonic and satisfies the conditions (a), (b) and (c) above. Then either $\phi$ is constant or $\phi$ is a totally geodesic map of constant rank 2 and constant energy density

A/B. Moreover, if $\tilde{M}$ denotes the universal cover of M, the map $\tilde{\phi}$ defined by the diagram factors

$$\begin{tikzcd} \tilde{M} \arrow[d] \arrow[dr, "\tilde{\phi}"] & \\ M \arrow[r, "\phi"] & N \end{tikzcd}$$

through a 2-sphere or a projective plane of constant curvature.

Remark

Eells-Lemaire [ 7 ] have noted that if $e(\phi) \le A/2B$ then $\phi$ must be constant.

The remainder of this section is taken up with the proof of this theorem.

To begin, choose a point $p \in M$ and diagonalise $(\phi^*h)_p$, i.e. write $(\phi^*h)_p = \sum_{i=1}^{m} \lambda_i\, \omega_i \otimes \omega_i$ with $g_p = \sum_{i=1}^{m} \omega_i \otimes \omega_i$ and $\lambda_1 \ge \lambda_2 \ge \ldots \ge 0$. Note that by the restriction on the rank of $\phi$ at most two of the $\lambda_i$'s are non-zero. With $|d\phi|_p^2 = \lambda_1 + \lambda_2$, $|\Lambda^2 d\phi|_p^2 = \lambda_1 \lambda_2$, the following inequality is self evident,

$$|\Lambda^2 d\phi|_p^2 \le \frac{|d\phi|_p^4}{4} \tag{2}$$

with equality if and only if $\lambda_1 = \lambda_2$.

4.2. LEMMA

Under the hypotheses of the theorem, if $\phi$ is not constant it is a totally geodesic map of rank 2 and constant energy density.

Remarks

(1) A totally geodesic map has constant rank because, as is easily seen, the kernel of its derivative is invariant under all parallel translations.

(2) A totally geodesic map always has constant energy density, for if $\phi$ is such a map then

$$de(\phi) = \langle \nabla d\phi, d\phi \rangle = 0.$$

Proof of 4 .2.

Under hypothesis (a) and using the inequality (2), the formula (1) yields

$$\begin{aligned}\Delta e(\phi) &= |\nabla^{\phi} d\phi|^2 + \langle d\phi(\mathrm{Ricci}^M v), d\phi(v)\rangle - R^N(d\phi(v), d\phi(w), d\phi(v), d\phi(w)) \\ &\geq |\nabla^{\phi} d\phi|^2 + A|d\phi|^2 - 2B|\Lambda^2 d\phi|^2 \\ &\geq |\nabla^{\phi} d\phi|^2 + |d\phi|^2 \left(A - \frac{B}{2}|d\phi|^2\right) \\ &\geq 0 \end{aligned} \tag{3}$$

this last inequality coming from the hypothesis that $e(\phi) \leq A/B$. Consequently $e(\phi)$ is subharmonic and by the divergence theorem is harmonic on the compact manifold M. All the inequalities in (3) can thus be replaced by equalities and using the third line the following information can be read off.

(i) $\nabla d\phi \equiv 0$ and so $\phi$ is totally geodesic

(ii) $|d\phi|^2 \left(A - \frac{B}{2}|d\phi|^2\right) \equiv 0$, and as $\phi$ is assumed not to be identically constant it can be deduced that $e(\phi) \equiv A/B$.

There remains to show that $\phi$ does not have rank 1. Suppose it did, then using (i) and (ii) together with the antisymmetry properties of the curvature tensor of $R^N$,

$$0 = \Delta e(\phi) \geq A|d\phi|^2 > 0$$

which is an evident contradiction. The proof of Lemma 4.2 is now complete.

In fact much more can be deduced with little extra work.

4.3. LEMMA

Under the same hypotheses the following hold.

(iii) $A|d\phi|^2 = \langle d\phi(\mathrm{Ricci}^M v), d\phi(v)\rangle$ at each point of M.

(iv) $2B|\Lambda^2 d\phi|^2 = R^N(d\phi(v), d\phi(w), d\phi(v), d\phi(w))$ at each point of M.

(v) At each $p \in M$, $\lambda_1 = \lambda_2 > 0$. In particular $\phi$ is horizontally homothetic.

Remark

A map $\phi:(M,g) \to (N,h)$ is *horizontally homothetic* if there is a constant $c > 0$ such that $h(d\phi(X), d\phi(Y)) = cg(X,Y)$ for all vectors X and Y orthogonal to ker $d\phi$. If $c = 1$ and $\phi$ is a submersion it is called a *Riemannian submersion.*

Proof of 4.3.

Since $\phi$ satisfies the conclusions of the lemma the following series of inequalities is valid.

$$\begin{aligned} 0 = \Delta e(\phi) &= \langle d\phi(\mathrm{Ricci}^M v), d\phi(v)\rangle - R^N(d\phi(v), d\phi(w), d\phi(v), d\phi(w)) \\ &\geq A|d\phi|^2 - 2B|\Lambda^2 d\phi|^2 \\ &\geq |d\phi|^2(A - \tfrac{B}{2}|d\phi|^2) \\ &= 0. \end{aligned}$$

In particular all inequalities are equalities. In view of (2), statement (v) follows from lines two and three, while (iii) and (iv) follow from lines one and two together with the hypotheses on the curvature of M and N.

As a result of this corollary the study of $\phi : M \to N$ is reduced to the study of a horizontally homothetic totally geodesic map. According to Vilms [33] there is a factorisation through a Riemannian manifold P,

$$\begin{tikzcd} M \arrow[d, "\psi"'] \arrow[dr, "\phi"] & \\ P \arrow[r, "\theta"'] & N \end{tikzcd} \tag{4}$$

where $\psi$ is totally geodesic submersion and $\theta$ is a totally geodesic immersion, indeed P is the manifold formed by collapsing the fibres of $\phi : M \to N$ and carries the induced metric. At this stage it is convenient to normalise the metric of M so that $e(\phi) \equiv 1$, this will make $\psi$ into a totally geodesic Riemannian submersion.

4.4. <u>LEMMA</u>

The manifold P has constant positive curvature.

Proof

Set $H = (\ker d\phi)$ , then $H$, the horizontal distribution, is integrable and so locally the diagram (4) takes the form

$$\begin{tikzcd}
V_1 \times V_2 \arrow[d, "\text{projection}"'] \arrow[dr, "\phi"] & \\
V_2 \arrow[r, "\theta"'] & U
\end{tikzcd}$$

where $V_1$ is vertical and $V_2$ is horizontal. Since $g$ is the product metric on $V_1 \times V_2$ (see [ 16,I,p.182]), the normalisation of $g$ together with (v) shows that for $X$, $Y$ tangent to $V_2$

$$\theta^*h(X,Y) = h(\theta_*X,\theta_*Y) = h(\phi_*(O,X),\phi_*(O,Y)) = g((O,X),(O,Y)).$$

In particular $\theta^*h$ agrees with the natural metric induced on $P$ by its definition. By its very construction $\theta:P \to N$ is totally geodesic while by (iv) the sectional curvatures of $N$ determined by planes in $d\phi(TP)$ are all equal to $B$. In particular the immersed totally geodesic submanifold $\phi(P)$ of $N$ has constant curvature $B$.

Remark

This lemma could have been proved by comparing the Ricci curvature of $P$ with that of $N$ in the horizontal directions, using (iii) of 4.3 and observing the fact that $P$ is a 2-manifold and so its Ricci curvature determines its curvature.

It should now be clear that the study of $\phi:M \to N$ can be reduced to the study of $\psi:M \to P$. Let $\pi:\tilde{M} \to M$ denote the universal cover of M and let $\tilde{\psi}:\tilde{M} \to P$ denote the composition $\tilde{M} \xrightarrow{\pi} M \xrightarrow{\psi} P$.

By de Rham's decomposition theorem [ 25 ] M can be written as a product of simply connected irreducible manifolds,

$$\tilde{M} = M_o \times M_1 \times \ldots \times M_n .$$

Clearly, one of these factors, $M_n$ say, is the universal cover of P, this means that

$$\tilde{M} = S^2 \times \tilde{Y}$$

where $\tilde{Y}$ is some simply connected manifold satisfying $\text{Ricci}^Y \geq B$, and $S^2$ is a Euclidean 2-sphere. It is a simple matter, now, to see that $\tilde{\psi}:S^2 \times Y \to P$ is one of the three following types,

$$\begin{tikzcd} S^2 \times \tilde{Y} \arrow[d] \arrow[dr] & \\ S^2 & S^2 \times Y \arrow[l] \end{tikzcd} \qquad \begin{tikzcd} S^2 \times \tilde{Y} \arrow[d] \arrow[dr] & \\ P^2 & S^2 \times Y \arrow[l] \end{tikzcd} \qquad \begin{tikzcd} S^2 \times Y \arrow[d] \arrow[dr] & \\ P^2 & P^2 \times Y \arrow[l] \end{tikzcd}$$

All the maps appearing are projections or covering maps or products of such maps. This completes the proof of Theorem 4 .1.

Remark

If the assumption rank $\phi \leq 2$ is replaced by max.rank $\phi = p$, $p \geq 3$ and $e(\phi) \leq A/B$ is replaced by $e(\phi) \leq \frac{p}{2(p-1)} \frac{A}{B}$ then a very similar result can be proved to the effect that $\tilde{\phi}:\tilde{M} \to N$ factors through a quotient of $S^p$ with a metric of constant curvature

(b) Maps from manifolds with boundary

This section is devoted to the study of the harmonic maps from manifolds with boundary as discovered by Hamilton [11]. The method is similar to that of the previous section, only formula (1) is needed in a integrated form. Specifically, if $\phi : M \to N$ is harmonic then

$$\int_M |\nabla d\phi|^2 + \langle d\phi(\mathrm{Ricci}^M(v)), d\phi(v)\rangle - R^N(d\phi(v), d\phi(w), d\phi(v), d\phi(w)) * 1$$

$$= \int_{\partial M} \langle \nabla_n d\phi,\ d\phi\rangle *_\partial 1 \qquad (5)$$

where n is the outward unit normal and $*_\partial 1$ is the volume element on the boundary of M with respect to the induced Riemannian structure.

In order to state the results a little of the geometry of $\partial M$ must be investigated. Let $E_1, \ldots, E_m$ be a frame field on M satisfying

(a) $E_1 \perp \partial M$ and $|E_1| \equiv 1$ on $\partial M$ and is outward pointing while $E_2, \ldots, E_m$ are tangent to $\partial M$.

(b) $[E_i, E_j] = 0$ for all $i,j = 1, \ldots, m$.

(c) If $e_i = (E_i)_p$ then $e_2, \ldots, e_m$ is an orthonormal basis of $T_p\partial M$.

If $\alpha$ is the second fundamental form of $\partial M$ then

$$\alpha(E_i, E_j) = P(\nabla_{E_i} E_j) \quad \text{on } \partial M \qquad (6)$$

where $P : TM|_{\partial M} \to (N\,\partial M)$ is orthogonal projection and the covariant derivation is that of $(M,g)$. Then $\alpha$ is a symmetric tensor field and relative to $E_1$ there is defined a symmetric linear transformation $A : T\partial M \to T\partial M$ according to the relationship

$$\langle \alpha(E_i,E_j),E_1\rangle = \langle AE_i,E_j\rangle. \tag{7}$$

4.5. LEMMA

The following identities hold,

(i) $Ae_i = -\nabla_{e_i}E_1$, in particular $\nabla_{e_i}E_1$ is tangent to $\partial M$.

(ii) $\text{Trace } A_p = \sum_{i=2}^{m} \langle \nabla_{e_i}E_i, e_1\rangle$

Proof

Using property (a) of the vector fields $E_i$, $i = 1,\ldots,m$,

$$0 = e_i \langle E_1,E_j\rangle = \langle \nabla_{e_i}E_1,E_j\rangle + \langle e_1,\nabla_{e_i}E_j\rangle.$$

From the characterisations of $\alpha$ and A in (6) and (7)

$$\langle -\nabla_{e_i}E_1,e_j\rangle = \langle \alpha(e_i,e_j),e_1\rangle = \langle Ae_i,e_j\rangle$$

But this is true for $j = 2,\ldots,m$, so

$$Ae_i = -\nabla_{e_i}E_1$$

and (i) is proved.

For (ii) note that a one hand trace $A_p = \sum_{i=2}^{m} \langle Ae_i,e_i\rangle$

while on the other

$$0 = e_i\langle E_i,E_1\rangle = \langle \nabla_{e_i}E_i,E_1\rangle + \langle e_i,\nabla_{e_i}E_1\rangle,$$

taking account of (6) and (7) using part (i) yields

$$\text{Trace } A_p = \sum_{i=2}^{m} \langle Ae_i,e_i\rangle = -\sum_{i=2}^{m} \langle \nabla_{e_i}E_1,e_i\rangle = \sum_{i=2}^{m} \langle \nabla_{e_i}E_i, E_1\rangle.$$

4.6. DEFINITION

The eigenvalues of $A_p$ are called the *principal curvatures* of $M$ at $p$.

convex if $A \leq 0$

Say that $\partial M$ is *totally geodesic if* $A \equiv 0$ and *minimal* if Trace $A = 0$. If Tr $A \leq 0$ say that $\partial M$ has *non-negative mean curvature*!

These ideas can now be applied to harmonic maps.

4.7. THEOREM

Suppose that $(M,g)$ is a compact Riemannian manifold with convex boundary and that $\phi:(M,g) \to (N,h)$ is harmonic and satisfies the Neuman condition $d\phi(n) = 0$ on $\partial M$. If $\mathrm{Ricci}^M \geq 0$ and $\mathrm{Riem}^N \leq 0$ then $\phi$ is totally geodesic. If either $\mathrm{Ricci}^M > 0$ somewhere or $\partial M$ is strictly convex somewhere then $\phi$ is constant. In any case, if $\partial M$ is minimal then $\phi|_{\partial M}:\partial M \to N$ is harmonic.

Proof

Let $p \in \partial M$ and consider vector fields $E_i$ $i = 1,\ldots,m$ as above. As $d(d\phi) = 0$ the relation

$$(\nabla_{E_i} d\phi)(E_j) = (\nabla_{E_j} d\phi)(E_i) \quad i,j = 1,\ldots,m \tag{8}$$

holds, while the hypothesis $d\phi(n) = 0$ translates to

$$d\phi(E_1) = 0. \tag{9}$$

Thus $$\langle \nabla_n d\phi, d\phi\rangle_p = \sum_{i=2}^{m} \langle (\nabla_{e_1} d\phi)(e_i), d\phi(e_i)\rangle$$

$$= \sum_{i=2}^{m} \langle (\nabla_{e_i} d\phi)(e_1), d\phi(e_i)\rangle \quad \text{by (8)}$$

expanding out and using (9) yields

$$\langle \nabla_n d\phi, d\phi\rangle_p = -\sum_{i=2}^{m} \langle d\phi(\nabla_{e_i} E_1), d\phi(e_1)\rangle. \tag{10}$$

Thus by the hypothesis on $\partial M$, $<\nabla_n d\phi, d\phi>_p \leq 0$ and this holds for each $p \in \partial M$. Combining this with the curvature restrictions and using formula (1) yields

$$0 \leq \int_M |\nabla d\phi|^2 + <d\phi(\text{Ricci}^M v), d\phi(v) - \text{Ricci}^N(d\phi(v), d\phi(\omega), d\phi(v) d\phi(\omega))$$

$$= \int_M <\nabla_n d\phi, d\phi>*_\partial| \leq 0.$$

Thus there is equality throughout and so both integrands vanish identically. For the integrand over M this means, in particular, that, $\nabla d\phi = 0$ and $\phi$ is totally geodesic.

If $\text{Ricci}^M > 0$ at some point then $d\phi$ must vanish there, by the constancy of $e(\phi)$ for the totally goedesic map $\phi$ the map must be constant.

On the other hand the integrand over $\partial M$ must also vanish identically and so by (10)

$$<d\phi(Ae_i), d\phi(e_i)> = 0.$$

Consequently if $A < 0$ at some point of $\partial M$ $d\phi$ must vanish there and so for the same reason as above $\phi$ must be constant.

The final statement is a direct application of Proposition 1.6.

As for solutions to the Dirichlet problem.

4.8. THEOREM.

Suppose $(M,g)$ is a compact Riemannian manifold with boundary and that $\phi:(M,g) \to (N,h)$ is harmonic and its restriction to $\partial M$ is constant. If $\mathrm{Ricci}^M \geq 0$, $\mathrm{Riem}^N \leq 0$ and $\partial M$ has non-negative mean curvature then $\phi$ is totally geodesic.* Further, if $\partial M$ is not minimal then $\phi$ is constant.

Proof

Let $E_1, \ldots, E_m$ be a frame field as described above. The harmonic equation at $p \in \partial M$ reads

$$-(\nabla_{e_1} d\phi)(e_1) = \sum_{i=2}^{m} (\nabla_{e_i} d\phi)(e_i). \tag{11}$$

Then
$$\begin{aligned} \langle \nabla_n d\phi, d\phi \rangle_p &= \langle (\nabla_{e_1} d\phi)(e_1), d\phi(e_1) \rangle \\ &= -\sum_{i=2}^{m} \langle (\nabla_{e_i} d\phi)(e_i), d\phi(e_1) \rangle \quad \text{by (11)} \end{aligned}$$

expanding out and using the fact that $d\phi(E_i) = 0$ $i = 2, \ldots, m$ on $\partial M$

$$\langle \nabla_n d\phi, d\phi \rangle_p = \sum_{i=2}^{m} \langle d\phi(\nabla_{e_i} E_i), d\phi(e_1) \rangle.$$

By the hypothesis on $\phi|\partial M$, and using Lemma 4.5(ii) the equality

$$\langle \nabla_n d\phi, d\phi \rangle_p = \mathrm{trace}\, A |d\phi(e_1)|^2 \tag{12}$$

can be seen.

Applying the curvature restrictions and the fact that $\mathrm{trace} A \leq 0$ yields as before

$$\begin{aligned} 0 &\leq \int_M |\nabla d\phi|^2 + \langle d\phi(\mathrm{Ricci}^M v), d\phi(v) - \mathrm{Riem}^N(d\phi(v), d\phi(\omega), d\phi(v), d\phi(\omega)) * 1 \\ &= \int_{\partial M} \langle \nabla_n d\phi, d\phi \rangle * 1 \leq 0 \end{aligned}$$

* This is still true if $\phi : \partial M \to N$ is merely assumed to be harmonic. If $\phi : \partial M \to N$ is constant $\phi$ maps $M$ to a geodesic of $N$.

and so equality holds throughout. As each integrand is non-negative they must vanish identically, in particular $\nabla d\phi = 0$ and $\phi$ is totally geodesic. From (12) trace $A|d\phi(n)|^2 = 0$ on $\partial M$ so if trace $A < 0$ somewhere, i.e. $\partial M$ is not minimal, then $d\phi(n) = 0$ there and so $|d\phi|^2 = 0$, by the constancy of $|d\phi|^2$ $\phi$ is constant.

(c) The index of harmonic maps

Let $\phi:(M,g) \to (N,h)$ be a harmonic map, if $v,w \in A(\phi^*TN)$ are variations of $\phi$ then

$$\nabla_v \nabla_w E(\phi) = \int_M \langle J_\phi v, w\rangle *1 = H_\phi(v,w)$$

where the second variation operator $J_\phi$ is a second order self-adjoint strongly elliptic operator given by

$$J_\phi v = -\mathrm{Tr}\nabla^2 v - \mathrm{Tr}R^N(d\phi, v)\, d\phi.$$

This calculation has been made in [32] and [22]. Mazet shows that the eigenvalues of $J_\phi$ are bounded below and thus the following definition can be made.

4.9. DEFINITION

The *index* of $\phi$, written index $\phi$, is the dimension of the largest subspace of $A(\phi^*TN)$ on which $H_\phi$ is negative definite. Note that index $\phi = 0$ if $\phi$ is a local minimum of the energy.

In the special case that $M = S^1$ a harmonic map is a closed geodesic in N and a well known theorem of Synge states that if N is even dimensional and orientable with $\mathrm{Riem}^N > 0$, then index $\Phi > 0$ for such a map. The method of proof is to construct a parallel normal field and then apply the form $H_\phi$ to it. Simons [31] has noted the corresponding result for the area functional, that for general harmonic maps takes the following form.

4.10. THEOREM

Suppose $\phi:(M,g) \to (N,h)$ is harmonic with $M$ compact and $\mathrm{Riem}^N > 0$. If $\phi$ admits a non-trivial parallel variation then either index $\phi > 0$ or $\phi(M)$ is contained in a geodesic of $N$.

Proof

Let $v \in A(\phi^*TN)$ be the supposed parallel variation. Then

$$H_\phi(v,v) = \int_M (|\nabla v|^2 - \mathrm{Tr}R^N(d\phi,v,d\phi,v)) *1.$$

The first term of the integrand is identically zero while, provided $\phi(M)$ does not lie in a goedesic of $N$, the second is strictly negative. In this case $H_\phi(v,v) < 0$ and the result is proved.

4.11. Examples

The hypothesis of the existence of a non-trivial parallel field is satisfied whenever $\phi(M)$ is contained in a submanifold admitting a parallel normal vector field.

In the paper of Simons cited, there appears at length a discussion of the second variation in particular with regard to minimal submanifolds of spheres. In fact the following is stated there. If $\dim M = p$ and $M$ is a minimal submanifold of $S^n$ then $\mathrm{index}_A M \geq n-p$ with equality if and only if $M$ is a totally geodesic sphere, $\mathrm{index}_A M$ denotes the index of $M$ with respect to the volume or area functional. However, the situation with regard to the energy functional is more complicated. Let $\Delta^M$ denote the Laplace-Beltrami operator on $(M,g)$; if $f:M \to \mathbb{R}$ is smooth

$$\Delta^M f = \mathrm{Tr}\nabla^2 f.$$

The eigenvalues of $\Delta^M$ are negative or zero, let them be

$$0 = \mu_o > -\mu_1 > -\mu_2 > \ldots \quad \text{with multiplicities } d_i.$$

4.12. PROPOSITION

Suppose that $\phi:(M,g) \to S^m$ is a non-constant harmonic map and let $\Phi:(M,g) \to S^n$ denote the composition of $\phi$ with the natural totally geodesic embedding $S^m \to S^n$. If $i = \max\{k;\ \mu_k \leq |d\phi|^2 \text{ and } \mu_k \not\equiv |d\phi|^2\}$ then

$$\text{index } \Phi \geq \text{index } \phi + (n-m) \sum_{k=0}^{i} d_k . \tag{13}$$

If $e(\phi)$ is constant then equality holds.

Proof

Let $I_n$ be the space of variations $w$ of $\Phi$ which are normal to $\phi$ and satisfy $H_\Phi(w,w) < 0$. To prove (13) it is sufficient to show that

$$\dim I_n \geq (n-m) \sum_{k=0}^{i} d_k. \tag{14}$$

For each k let $\{g_j^k; j = 1,\ldots,d_k\}$ be a complete set of eigenfunctions for the eigenvalue $-\mu_k$ of $\Delta^M$, then

$$\Delta^M g_j^k = -\mu_k g_j^k \quad j = 1,\ldots,d_k.$$

Also let $\{u_\alpha; d = 1,\ldots,n-m\}$ be mutually pointwise orthogonal parallel sections of $N(S^m)$, the normal bundle of $S^m$ in $S^n$, of unit length. Then, regarding these as variations of $\Phi$ and letting R denote variously the curvature tensor of $S^m$ and $S^n$ which satisfies $R(u,v)w = \langle u,w\rangle v - \langle u,v\rangle w$;

$$J_\Phi (g_j^k u^\alpha) = -\mathrm{Tr}\, \nabla^2 g_j^k u^\alpha - g_j^k \mathrm{Tr}(d\Phi, u^\alpha)d\Phi$$

$$= -\Delta^M g_j^k u^\alpha - g_j^k |d\phi|^2 u^\alpha \qquad (15)$$

$$= (\mu_k - |d\phi|^2) g_j^k u^\alpha.$$

Thus,

$$H_\Phi(g_j^k u^\alpha, g_j^k u^\alpha) = \int_M (\mu_k - |d\phi|^2)|g_j^k|^2 * 1 < 0 . \qquad (16)$$

Therefore the variations $\{g_j^k u^\alpha;\ k=0,\ldots,i, j = 1,\ldots,d_k, \alpha = 1,\ldots,n-m\}$ are independent and contained in I ; (14), and consequently (13), is proved.

Now suppose that $e(\phi)$ is constant, to show that equality holds in (13) it is sufficient to demonstrate that the following two statements are true:

(a) If w is a variation of $\Phi$ satisfying $J_\Phi w = -\lambda w$ for some $\lambda > 0$ and if $w^N$ denotes the normal part of w then $J_\Phi w^N = -\lambda w^N$.

(b) $\mathrm{Dim}\ I_n = (n-m) \sum_{k=0}^{i} d_k$.

So suppose w satisfies the hypothesis of (a), setting $w^T + w^N = w$ and using the linearity of $J_\phi$,

$$J_\phi\, w^T = -\lambda w - J_\phi\, w^N. \tag{17}$$

Taking normal parts of this equation shows that

$$-\lambda w^N = (J_\phi w^N)^N$$

while a direct calculation involving the fact that $S^m$ is totally geodesic in $S^n$ shows that $J_\phi w^N$ is actually normal to $S^m$ so the conclusion of (a) holds.

For (b), the fact that the set $\{u_\alpha : \alpha = 1,\ldots,n-m\}$ is independent and spans the normal space at each point shows that (b) need only be proved in the case $n-m = 1$. So suppose $n-m = 1$ and relabel $u = u_1$. If $w \in I$ then $w = fu$ for some function $f: M \to \mathbb{R}$,

so that

$$\begin{aligned} J_\phi w &= -\mathrm{Tr}\nabla^2 fu - f\mathrm{Tr}R(d\Phi,u)d\Phi \\ &= -\Delta^M f.u - |d\phi|^2.fu. \end{aligned}$$

If, as may be supposed, $J_\phi w = -\lambda w$ for some $\lambda > 0$ then f satisfies the equation

$$\Delta^M f = (\lambda - |d\phi|^2)f. \tag{18}$$

Thus the constant $(\lambda - |d\phi|^2)$ is subject to the restrictions $\lambda - |d\phi|^2 \le 0$ and $\lambda - |d\phi|^2 = -\mu_k$ for some k. But $0 < \lambda = |d\phi|^2 - \mu_k$ if and only if $k \in \{1,\dots,i\}$. As a consequence, equation (18) has $\sum_{k=0}^{i} d_k$ linearly independent solutions so

$$\dim I_n = \sum_{k=0}^{i} d_k$$

and (b) and the proposition are proved.

4.13. <u>Remarks and Calculations</u>

The remarks which follow give two examples which show that equality may hold in (13) but this is not always the case. The first of the two gives a contrast between the energy and the volume functionals for maps between spheres.

(i) Let $i_m^n : S^m \to S^n$ be the standard totally geodesic embedding. If $id_m : S^m \to S^m$ is the identity map then $e(id_m) \equiv m/2$ so that $i = 0$. (The first few eigenvalues of the Laplace-Beltrami operator on $S^m$ are $0, -m, -2(m+1)$, [ 4 ]). The proposition implies

$$\text{index } i_m^n = \text{index } id_m + (n-m).$$

However, Smith [32] has computed that $\text{index } id_m = \begin{cases} m+1 & \text{if } m \ge 3 \\ 0 & \text{if } m = 2 \end{cases}$

so that

$$\text{index } i_m^n = \begin{cases} n+1 & \text{if } m \ge 3 \\ n-2 & \text{if } m = 2 \end{cases}$$

(ii) If $i = 0$ and $\dim I = n-m$ then $e(\phi)$ is constant.

For then $I$ is spanned by $\{u^\alpha; \alpha = 1,\ldots,n-m\}$ and so the only non-zero solutions to (18) are $f$ = constant and then $0 = \lambda - |d\phi|^2$.

(iii) Suppose $M$ is a compact orientable Riemannian surface admitting a branched covering $\phi: M \to S^2$. Then $i = 0$ and $e(\phi)$ is non-constant, so by (ii) $\dim I > n-2$ and

$$\text{index } \Phi \;>\; \text{index } \phi + n-2.$$

This means, in particular that there is a non-trivial solution of equation (18) on $M$.

# CHAPTER 5

## HARMONIC MAPS OF SURFACES

### (a) Introduction

In the paper [30] Shibata tackled the question of whether there exists a harmonic homeomorphism homotopic to a given homeomorphism between two closed Riemann surfaces. However, there are several errors in his paper which are of varying degrees of importance. For example his proof of Lemma 3 is incorrect even though the statement is correct. The present author has discovered other errors pertaining to, for example, Shibata's class of differentials, in fact the proof of Lemma 5 contains a serious error concerning the total variation of the limit of a sequence of functions.

Nevertheless the cited paper contains a number of good ideas and the present author felt that if the errors could be corrected then Shibata's final conclusion could be re-established.

In order to fix ideas, let M, N be two closed Riemann surfaces of equal genus

$$\text{genus}\ (M) = \text{genus}\ (N) \geq 1$$

and let $\psi : M \to N$ be a quasiconformal homemorphism. Suppose h is a conformal Rieman metric on M, this means that isothermal charts for h are also complex coordinates for the given complex structure on N. The question to be answered is:

Q. Does there exist a harmonic map $\psi:M \to (N,h)$ which is both homotopic to $\psi$ and is a homeomorphism?

Combining the existence results of Lemaire [18] with the methods of Sampson [27], Schoen-Yau [29] and Hartman [12], a partial answer can be given.

5.1. <u>THEOREM</u>

There always exists a harmonic map $\phi:M \to (N,h)$ homotopic to $\psi:M \to N$. If h has non-positive Gaussian curvature then all such harmonic maps are diffeomorphisms. Moreover, if h has strictly negative Gaussian curvature the map $\phi$ is the unique harmonic map homotopic to $\psi$.

Now let H(M,N) be the class of maps $\phi:M \to N$ satisfying the following hypotheses.

(i) $\phi$ is a homeomorphism homotopic to $\psi$.

(ii) $\phi$ together with $\phi^{-1}$ has locally square integrable derivatives in any, and hence all, local complex coordinates.

(iii) $\phi$ and $\phi^{-1}$ are absolutely continuous in the two dimensional sense, i.e. $\phi$ and $\phi^{-1}$ map null sets to null sets.

The idea of Shibata's 'solution' to Q is as follows.

Let $\Omega$ be some class of conformal Riemannian metrics on M together with a decomposition by a real parameter into compact subsets, $\Omega = \bigcup_A \Omega_A$ say, with $\Omega_A$ compact. For each $g \in \Omega_A$ a subclass $H_g$ of H(M,N) is defined as those $\phi \in H(M,N)$ satisfying

(iv) $E(\phi^{-1},g) = \int_N (|z_w|^2 + |z_{\bar{w}}|^2)\rho^2(z)\, dudv \le A(K + 1/K)$

where K is the maximal dilatation of $\psi$, $w = u + iv$ is a local complex coordinate on N and $g = \rho^2(dx^2 + dy^2)$ with respect to the local complex coordinate $z = x + iy$ on M.

It is then shown that $E(\phi)$ attains its minimum on $H_g$ at the map $\phi_g$ say. The resulting map $\Omega_A \to \mathbb{R}$ defined by $g \to E(\phi_g)$ is then minimised within the class $\Omega_A$ at the metric $g_A$ and associated map now called $\phi_A$. The final step of the proof should be to show that $\phi_A$ is harmonic when A is sufficiently large. However, the present author believes that Shibata makes another error here and has himself been unable to resolve the situation.

The purpose of this chapter then, is to describe what parts of Shibata's paper can be revived and to point out the remaining problems.

## (b) A compact family of metrics

One way to construct such a family of metrics is to make use of convergence properties of holomorphic functions on the unit disc. Let

$$U = \{z \in \mathbb{C};\ |z| < 1\}$$

and let $H^2$ be the set of holomorphic functions on U whose Taylor series about zero are square summable, so if

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \text{ in } U$$

then $f \in H^2 \iff \sum_{n=0}^{\infty} |a_n|^2 < \infty$. This last expression gives the square of the $H^2$ norm of f and $H^2$ becomes a Hilbert space with respect to the associated innerproduct. For the properties of the space $H^2$ the reader is referred to [13]. The other space of interest in this section is denoted $L_H^2$ and comprises the holomorphic functions $f:U \to \mathbb{C}$ satisfying

$\frac{1}{\pi} \int_U |f|^2 \, dxdy < \infty$, $L_H^2$ becomes a Hilbert space in the inner product associated to this norm [34].

5.2. <u>DEFINITION</u>

A continuous map $j:H \to E$ between Hilbert spaces is called *compact* if $j(B)$ is relatively compact in E for all bounded subsets $B \subset H$.

5.3. <u>THEOREM</u>

The natural inclusion $j:H^2 \to L_H^2$ is compact.

<u>Proof</u>

The map $j:H^2 \to L_H^2$ is given by $f \to f$, indeed this is clearly continuous since if $f(z) = \sum_{n=0}^{\infty} a_n z^n$ then

$$|f|_2 = \left(\sum_{n=0}^{\infty} |a_n|^2\right)^{\frac{1}{2}} \text{ is the } H^2 \text{ norm of } f$$

while $\|f\| = \left(\frac{1}{\pi} \int_U |f|^2 \, dxdy\right)^{\frac{1}{2}} = \left(\sum_{1}^{\infty} \frac{|a_n|^2}{n+1}\right)^{\frac{1}{2}}$ is the $L_H^2$ norm of f.

Thus $f \in H^2 \quad \|f\| \le |f|_2$ and $j$ is continuous.

To prove the compactness of $j$ it is clearly sufficient to show that the $H^2$ unit ball is relatively compact in $L^2_H$ or equivalently that for any $\varepsilon > 0$ there is an $\varepsilon$-net for this ball in the $L^2_H$ metric.

Let $\varepsilon > 0$ and set $F = \{n \in \mathbb{N};\ \frac{1}{n+1} > \varepsilon^2/2\}$ so that $|F| < \infty$. The set of power series $\sum_{n=0}^{\infty} a_n z^n$ with $a_n = 0$ for $n \notin F$ is finite dimensional so its unit ball in any norm is relatively compact. Let $f_1,\dots,f_m$ be an $\frac{\varepsilon}{\sqrt{2}}$ -net for the metric of $L^2_H$.

If $f = \sum_{n=0}^{\infty} a_n z^n$ satisfies $|f|_2 < 1$ then

$$\sum_{n\in F} \frac{|a_n|^2}{n+1} \le \sum_{n=0}^{\infty} |a_n|^2 < 1$$

so $\exists\ i \in \{1,\dots,m\}$ such that $$\sum_{n\in F} \frac{|a_n - a_n^i|^2}{n+1} < \varepsilon^2/2 \qquad (1)$$

where $f_i = \sum_{n=0}^{\infty} a_n^i z^n$ and $a_n^i = 0$ for $n \notin F$.

Thus $$\sum_{n\notin F} \frac{|a_n - a_n^i|^2}{n+1} = \sum_{n\notin F} \frac{|a_n|^2}{n+1} \le \frac{\varepsilon^2}{2} \sum_{n\notin F} |a_n|^2$$

this last inequality following from the definition of $F$. Using the fact that $\sum_{n=0}^{\infty} |a_n|^2 < 1$ it can be seen that

$$\sum_{n \notin F} \frac{|a_n - a_n^i|^2}{n+1} \leq \frac{\varepsilon^2}{2} . \tag{2}$$

(1) and (2) now show that

$$\sum_{n=0}^{\infty} \frac{|a_n - a_n^i|^2}{n+1} \leq \varepsilon^2$$

showing that $f_1, \ldots, f_m$ is an $\varepsilon$-net for the unit ball of $H^2$ in the $L^2_H$ metric. The proof is complete.

5.4. <u>COROLLARY</u>

If $\{f_i\}_{i=1}^{\infty}$ is a bounded sequence in $H^2$ then there exists a subsequence, still called $f_i$, and an $f \in H^2$ such that $f_i \to f$ weakly in $H^2$ and strongly in $L^2_H$. In particular, $f_i \to f$ uniformly on compact subsets of $U$.

<u>Proof</u>

Since $H^2$ is a Hilbert space there is a subsequence, still called $f_i$, and an $f \in H^2$ such that $f_i \to f$ weakly in $H^2$. By Theorem 5.3 it can be assumed that there is an $\tilde{f} \in L^2_H$ such that, for this subsequence, $f_i \to \tilde{f}$ in $L^2_H$ and by well known theorems on $L^2_H$, [34], this convergence is uniform on compact subsets of $U$. To complete the proof it suffices to show that $f = \tilde{f}$. Let the Taylor expansions of the functions in question be

$$f_i(z) = \sum_{n=0}^{\infty} a_n^i z^n, \quad f(z) = \sum_{n=0}^{\infty} a_n z^n$$

and $$\tilde{f}(z) = \sum_{n=0}^{\infty} \tilde{a}_n z^n.$$

It suffices to show that $a_n = \tilde{a}_n$ for all values of n. By the weak convergence of the $f_i$,

$$a_n^i = \langle f_i, z^n\rangle \to \langle f, z^n\rangle = a_n \quad \text{for each } n. \qquad (3)$$

While the uniform convergence implies, with superscripts denoting differentiation,

$$a_n^i = \frac{1}{n!} f_i^n(0) \to \frac{1}{n!} \tilde{f}^n(0) = \tilde{a}_n \text{ for each } n. \qquad (4)$$

From (3) and (4) $a_n = \tilde{a}_n$ for each n as required.

A family of conformal metrics will be defined on M using elements of $H^2$ and the compactness of this family will be deduced from 5.4. Let $D_1, \ldots, D_k$ be complex coordinates discs covering M, with local parameters $z_1, \ldots, z_m$, these subscripts will often be dropped, it is hoped, without causing confusion. Let $\lambda_1, \ldots, \lambda_m$ be a smooth partition of unity subordinate to this covering. If $f_i$, $1 \leq i \leq m$ are given functions on U, construct in each $D_i$ the tensor field

$$h_i = |f_i|^2 (dx^2 + dy^2), \qquad z = x + iy$$

and on M set $g = \sum_{i=1}^{m} \lambda_i h_i$.

Suppose each $f_i$ satisfies, for some fixed $A > 1$,

(a) $f_i \in H^2$ and has no zeros in U.

(b) $|f_i|_2 \leq A$

(c) $\int_U |f_i|^2 \, dxdy \geq c_A$, for some suitable constant depending on A.

(d) $\sum_{i=1}^{k} \frac{1}{\pi} \int_U \lambda_i |f_i|^2 \, dxdy = 1$

then say $g \in \Omega_A$ and use this as a defining condition for $\Omega_A$.

The success of the method relies on the family $\Omega_A$ being a compact family of conformal metrics on M.

5.5. PROPOSITION

For each $A > 1$ there is a value of $c_A > 0$ such that the family $\Omega_A$ is a non-empty compact family of positive definite conformal metrics on M.

Proof

That $\Omega_A$ is non-empty is clear, for the $f_i$'s may all be chosen as constants and a possible choice of $c_A$ is then clear. To see that any $g \in \Omega_A$ is positive definite note that each $g_i$ is on $D_i$ and that at each point g is a convex combination of the $g_i$, g is clearly a conformal metric. To see that $\Omega_A$ is compact let $\{g_j\}_{j=1}^{\infty}$ be a sequence from $\Omega_A$, with $g_j = \Sigma \lambda_i |f_{ij}|^2 (dx^2 + dy^2)$

then using 5.4 find subsequences of the $f_{ij}$ converging as in 5.4. Suppose $f_{ij} \to f_i$, then $g_i \to \Sigma\lambda_i |f_i|^2(dx^2 + dy^2)$ uniformly together with all its derivatives. There remains to check that $g = \Sigma\lambda_i |f_i|^2(dx^2 + dy^2) \in \Omega_A$. Firstly note that by the weak convergence $f_{ij} \to f_i$ for each $i = 1,\dots,k$

$$|f_i|_{H^2} \le \liminf_{j\to\infty} |f_{ij}|_{H^2} \le A \quad \text{so (b) holds.}$$

Since $f_{ij} \to f_i$ strongly in $L^2_H$, $\int_U |f_i|^2 \, dxdy = \lim \int_U |f_{ij}|^2 dxdy \ge c_A$

so (c) is satisfied, similarly so is (d). Finally, since each $f_{ij}$ is never zero in U, $f_i$ is either never zero or is identically zero by the theorem of Hurwitz, but $f_i$ satisfies (c) and so this second alternative is impossible, thus (a) is satisfied.

(c) <u>The Variational Problems</u>

Returning, now, to the class $H_g$ of homeomorphism defined by (i) - (iv) it will be useful to consider in more detail their differentiability and measure theoretic properties.

Let $z = x + iy$ and $w = u + iv$ be uniformizing parameters on the universal cover of M and N respectively, suppose $\phi \in H_g$ and that $\tilde{\phi}$ is a lift of $\phi$. Define the complex vector fields

$$\partial/\partial z = \tfrac{1}{2}(\partial/\partial x - i\partial/\partial y)$$

$$\partial/\partial \bar{z} = \tfrac{1}{2}(\partial/\partial x + i\partial/\partial y)$$

so that the Jacobian of $\tilde{\phi}$ is given by $|\frac{\partial\tilde{\phi}}{\partial z}|^2 - |\frac{\partial\tilde{\phi}}{\partial\bar{z}}|^2$ . If $\tilde{\phi}$ is represented by $z \to w$ this may be written $|w_z|^2 - |w_{\bar{z}}|^2$.

5.6. PROPOSITION

If $\phi$ satisfies (i) - (iii) then $\tilde{\phi}$ is totally differentiable almost everywhere. Moreover if B is a Borel set

$$\int_B (|w_z|^2 - |w_{\bar{z}}|^2)dxdy = \int_{\tilde{\phi}(B)} dudv.$$

The same results hold for $\tilde{\phi}^{-1}$.

Proof

The existence of the partial derivatives is standard and can be found in [23] or [ 2], the existence of the total derivative is due to Gehring and Lehto, their proof is reproduced in [2 ].

Now let $\varepsilon > 0$ and find $F \subset B$ closed with the property that $w_z$ and $w_{\bar{z}}$ exist and are continuous in F and that

$$\max \left\{ \int_{\tilde{\phi}(B-F)} dudv, \int_{B-F} |w_z|^2 + |w_{\bar{z}}|^2 dxdy \right\} < \varepsilon.$$

This can be done using the absolute continuity properties of $\phi$.

Since $\int_{\phi(F)} dudv = \int_F (|w_z|^2 - |w_{\bar{z}}|^2)dxdy$ (see [ 26 ]) the following holds,

$$\int_{\phi(B)} dudv - \int_B (|w_z|^2 - |w_{\bar{z}}|^2)dxdy \leq \int_{\phi(B-F)} dudv + \int_{B-F} (|w_z|^2 + |w_{\bar{z}}|^2)dxdy$$
$$< 2\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary the result is proved.

The first variational problem of Shibata can now be stated and solved.

Problem

Minimise E within the class $H_g$ for each $g \in \Omega_A$.

The solution to this problem uses the direct method together with the lower semi-continuity of E and the following beautiful result of J. Lelong-Ferrand [17].

5.7. THEOREM

Suppose M and N are closed orientable surfaces of the same genus and not homeomorphic to a sphere. Let h be a Riemannian metric on N, then the set $\{\phi:M \to N: \phi \text{ is a homeomorphism}, E(\phi) < C\}$ is uniformly equicontinuous for each $C > 0$.

As a first step towards solving problem Q let $\phi_n$ be a minimising sequence for E in $H_g$. By condition (iv) and 5.7 both $\phi_n$ and $\phi_n^{-1}$ form equicontinuous families and so it may be assumed that $\phi_n \to \phi$ and $\phi_n^{-1} \to \phi^{-1}$ uniformly to produce a homeomorphism which is a candidate for a minimum of E over $H_g$. There remains to show that $\phi \in H_g$.

5.8. <u>PROPOSITION</u>

The maps $\phi$ and $\bar{\phi}^{1}$ have locally square integrable derivatives and satisfy $E(\phi) \leq \liminf_{n \to \infty} E(\phi_n)$

$$\text{and} \quad E(\phi^{-1},g) \leq \liminf_{n \to \infty} E(\phi_n^{-1},g).$$

<u>Proof</u>

It is clearly sufficient to prove those statements concerning $\phi$, for those concerning $\phi^{-1}$ follow in the same way. The following will be demonstrated.

<u>Claim</u>

If $\phi_n$ is a sequence of functions on U with locally square integrable derivatives satisfying $\int_U |d\phi_n|^2 \, dxdy \leq L^2$, for some L, and if $\phi_n \to \phi$ uniformly on U, then $\phi$ has locally square integrable derivatives and $d\phi_n \to d\phi$ weakly in $L^2(U)$. This is clearly sufficient to prove the proposition once it has been noted that the norm on a Hilbert space is lower semi-continuous in the weak topology.

Proof of claim:

Let $\psi$ be a smooth function with compact support in U.

Then $0 = \int_{\partial U} \psi(\bar{\phi}_n - \bar{\phi}_m)dy = \int_U \{ \frac{\partial\psi}{\partial x} (\bar{\phi}_n - \bar{\phi}_m) + \psi(\frac{\partial\bar{\phi}_n}{\partial x} - \frac{\partial\bar{\phi}_m}{\partial x})\}dxdy$

using Stokes theorem.

Thus, if $\varepsilon > 0$ and n, m are sufficiently large,

$$\int_U \psi(\frac{\partial\bar{\phi}_n}{\partial x} - \frac{\partial\bar{\phi}_m}{\partial x})dxdy < \varepsilon$$

so the sequence $\int_u \psi \frac{\partial\bar{\phi}_n}{\partial x} dxdy$ is Cauchy and has a limit which will be called $T(\psi)$, T is clearly linear in $\psi$. Moreover T extends to a continuous linear functional $L^2(u) \to \mathbb{C}$ since

$$|T(\psi)| = \lim_n \int_u \psi \frac{\partial\bar{\phi}_n}{\partial x} dxdy| \leq L(\int_u |\psi|^2 dxdy)^{\frac{1}{2}}.$$

Thus, by the Riesz representation theorem, there is $\phi_\infty \in L^2(u)$ satisfying $T(\psi) = \int_U \psi\bar{\phi}_\infty dxdy$ and the relation

$\int_u \psi\frac{\partial\bar{\phi}_n}{\partial x} dxdy \to \int_u \psi\bar{\phi}_\infty dxdy$ for each $\psi$ means that $\phi_\infty$ is the weak limit of $\frac{\partial\phi_n}{\partial x}$. That $\phi_\infty$ is the derivative of $\phi$ with respect to x follows from the identities

$$\int_U \psi\bar{\phi}_\infty dxdy = \lim_n \int_U \psi \frac{\partial\bar{\phi}_n}{\partial x} dxdy$$

$$= - \lim_{n} \int_U \frac{\partial \psi}{\partial x} \bar{\phi}_n \, dxdy$$

$$= - \int_U \frac{\partial \psi}{\partial x} \bar{\phi} \, dxdy.$$

The proof that $\frac{\partial \phi}{\partial y}$ exists and is the weak $L^2$ limit of $\frac{\partial \phi_n}{\partial y}$ follows the same lines.

5.9. PROPOSITION

The maps $\phi$ and $\phi^{-1}$ have the absolute continuity properties expressed in condition (iii).

Proof.

Again this will only be proved for the map $\phi$, the idea is due to Ahlfors [ 1 ]. The calculation which follows takes place in the universal covers with parameters as above. Let D be a small disc then it is easy to see that

$$\int_{\phi(D)} dudv = \lim_{n\to\infty} \int_{\phi_n(D)} dudv \tag{5}$$

using the continuity properties of Lebesgue measure and the convergence properties of the $\phi_n$.

By Proposition 5.6.

$$\int_{\phi_n(D)} dudv = \int_D (|\frac{\partial w_n}{\partial z}|^2 - |\frac{\partial w_n}{\partial \bar{z}}|^2 \, dxdy. \tag{6}$$

To handle the convergence of these quantities define a double sequence of functions,

$$A_{mn}(r) = -\frac{1}{2i}\int_{|z-z_o|=r} (w_m - w_n)\, d\overline{(w_m - w_n)} \tag{7}$$

for almost all $r < r_o$ say. By approximating by $C^2$ functions and using Stoke's theorem it can be seen that

$$A_{mn}(r) = \int_{|z-z_o|\le r} \left(\left|\frac{\partial w_m}{\partial z} - \frac{\partial w_n}{\partial z}\right|^2 - \left|\frac{\partial w_m}{\partial \bar z} - \frac{\partial w_n}{\partial \bar z}\right|^2\right) dxdy. \tag{8}$$

Let $\varepsilon > 0$ then for m,n sufficiently large

$$A_{mn}(r) \le \varepsilon \int_{|z-z_o|=r} \left|\frac{\partial w_m}{\partial z}\right| + \left|\frac{\partial w_m}{\partial \bar z}\right| + \left|\frac{\partial w_n}{\partial z}\right| + \left|\frac{\partial w_n}{\partial \bar z}\right| \, d\theta \tag{9}$$

where $z = z_o + re^{i\theta}$, and this holds for almost all $0 < r < r_o$.

In view of (9) and since the $E(\phi_n)$ are uniformly bounded there is a positive constant C independent of n such that

$$\int_0^{r_o} [A_{mn}(r)]^2 \, dr \le \varepsilon C \text{ for sufficiently large m,n}$$

in particular $\lim\limits_{m,n\to\infty} \int_0^{r_o} [A_{mn}(r)]^2 dr = 0.$ (10)

On the other hand

$$A_{mn}(r) = \int_{|z-z_o|\le r} \left(\left|\frac{\partial w_n}{\partial z}\right|^2 - \left|\frac{\partial w_m}{\partial \bar z}\right|^2\right) dxdy + \int_{|z-z_o|\le r} \left(\left|\frac{\partial w_n}{\partial z}\right|^2 - \left|\frac{\partial w_m}{\partial \bar z}\right|^2\right) dxdy$$

$$-2\mathrm{Re} \int_{|z-z_o|\le r} \left(\frac{\partial w_m}{\partial z} \left(\overline{\frac{\partial w_n}{\partial z}}\right) - \frac{\partial w_m}{\partial \bar z} \left(\overline{\frac{\partial w_n}{\partial \bar z}}\right)\right) dxdy.$$

Letting $m \to \infty$ and then $n \to \infty$

$$\lim_{n\to\infty} \lim_{m\to\infty} A_{mn}(r) = 2\int_{\tilde{\phi}(D_r)} dudv - 2\int_{D_r} \left(\left|\frac{\partial w}{\partial z}\right|^2 - \left|\frac{\partial w}{\partial \bar{z}}\right|^2\right)dxdy \qquad (11)$$

by (5) and the weak convergence of the complex derivatives. Here $D_r = \{Z; |Z-Z_o| \leq r\}$.

From (8) it can be seen that the $A_{mn}$ are uniformly bounded so Lebesgue's dominated convergence theorem can be applied to (10) to obtain

$$O = \lim_{n\to\infty} \lim_{m\to\infty} \int_o^{r_o} [A_{mn}(r)]^2 dr = \int_o^{r_o} \lim_{n\to\infty} \lim_{m\to\infty} [A_{mn}(r)]^2 dr.$$

So that the expression on the right hand side of (11) vanishes for almost all $r \in (O, r_o)$, by continuity it vanishes for all r. Consequently

$$\int_{\tilde{\phi}(D_r)} dudv = \int_{D_r} \left(\left|\frac{\partial w}{\partial z}\right|^2 - \left|\frac{\partial w}{\partial \bar{z}}\right|^2\right)dxdy$$

for all $r \in (O, r_o)$, which is sufficient to prove the proposition.

The content of the previous two propositions is that the limit map $\phi$ is a member of $H_g$. Consequently the following theorem may be stated.

5.10. THEOREM

The first variational problem has a solution. Notice that nothing has been said about the uniqueness of this solution.

For each $g \in \Omega_A$ let $\phi_g$ be a solution to the first variational problem. The second variational problem may be stated.

<u>Problem</u>

Minimise E over the class of maps $\{\phi_g; g \in \Omega_A\}$.

In effect the problem is to minimise the map $\Omega_A \to \mathbb{R}_{\geq 0}$ given by $g \to E(\phi_g)$, however in approaching the problem from this viewpoint some information about the associated maps is lost. The method following is directly parallel to that of Shibata's and uses the information obtained in the solution of the first variational problem.

Let $\phi_n$ be a minimising sequence for E in the class $\{\phi_g; g \in \Omega_A\}$ and let $g_n$ be the associated metrics. Using 5.5 and 5.7 and condition (iv) there can be found a subsequence, a homeomorphism $\phi$ and a metric $g_\infty$ such that $\phi_n \to \phi, \phi_n^{-1} \to \phi^{-1}$ and $g_n \to g_\infty$ uniformly. By following the arguments used above it can be seen that $\phi$ satisfies (i) - (iii), there remains to show that (iv) holds in order to see that $\phi \in H_{g_\infty}$.

5.11. <u>PROPOSITION</u>

The inequality

$$E(\phi^{-1}, g\ ) \leq \liminf_{n \to \infty} E(\phi_n^{-1}, g_n).$$

is valid and hence $\phi \in H_{g_\infty}$

Proof.

For each n write $g_n = f_n g_\infty$ where $f_n : M \to \mathbb{R}_{>0}$ is smooth.

Note that $f_n \to 1$ uniformly on M. Denote by $e(\phi_n^{-1}, g_n)$ the energy density of $\phi_n^{-1} ; (N,h) \to (M,g_n)$ and by $e(\phi_n^{-1}, g_\infty)$ that of $\phi_n^{-1} : (N,h) \to (M,g_\infty)$ so that $e(\phi_n^{-1}, g_n) = f_n(\phi_n^{-1}) e(\phi_n^{-1}, g_\infty)$

and $E(\phi_n^{-1}, g_n) = \int_N f_n(\phi_n^{-1}) e(\phi_n^{-1}, g_\infty) * 1.$

Let $\varepsilon > 0$ and choose $n_o$ such that if $n > n_o$ then $f_n(\phi_n^{-1}) > 1-\varepsilon$ on N.

With this being the case

$$E(\phi_n^{-1}, g_n) > \int_N e(\phi_n^{-1}, g_\infty) * 1 - \varepsilon \int_N e(\phi_n^{-1}, g_\infty) * 1.$$

Consequently, $\inf\limits_{n > n_o} E(\phi_n^{-1}, g_n) \geq (1-\varepsilon) \inf\limits_{n > n_o} E(\phi_n^{-1}, g_\infty)$

so that $\forall\ \varepsilon > 0, \liminf\limits_{n \to \infty} E(\varepsilon_n^{-1}, g_n) \geq (1-\varepsilon) \liminf\limits_{n \to \infty} E(\phi_n^{-1}, g_\infty).$

Together with the relation

$$E(\phi^{-1}, g_\infty) \leq \liminf_{n \to \infty} E(\phi_n^{-1}, g_\infty)$$

which has been used before, this furnishes the result.

With this proposition proved the maps $\phi$ and $\phi_g$ can be compared as follows

$$E(\phi_{g_\infty}) \leq E(\phi) = \inf \{E(\phi_g); g \in \Omega_A\}.$$

The inequality follows from the fact that $\phi \in H_{g_\infty}$, while $\phi_{g_\infty}$ solves the first variational problem for $g_\infty$. The equality follows by the construction of $\phi$.

5.12. THEOREM

The second variational problem has a solution.

(d) The Variational Equation

The principal aim is to show that one of these minimising maps is harmonic. Recall from Lemma A after Theorem 2.17 that if $\phi: M \to (N,h)$ is harmonic then $(\phi^*h)^{2,0}$ is a holomorphic quadratic differential on M. This is the definition Shibata works with.

5.13. DEFINITION

Suppose $\phi: M \to (N,h)$ is continuous and has locally square integrable derivatives, say that $\phi$ is S-*harmonic* or *harmonic in the sense of Shibata* if $(\phi^*h)^{2,0}$ is holomorphic. The connection between this definition and the energy functional is given in the following result.

5.14. PROPOSITION

Suppose that $\phi$ is the uniform limit of a sequence of elements of $H(M,N)$. If $\phi$ minimises E over the class of all such maps then $\phi$ is S-harmonic.

Proof

Let $|z| < 1$ be a local parameter on M and let $\lambda : U \to \mathbb{C}$ be smooth with compact support. If $|\zeta| < 1$ represents the same chart as $|z| < 1$ then the maps given locally by $\zeta = z + \varepsilon\lambda(z)$, generate diffeomorphisms of M for $\varepsilon$ sufficiently small, call these maps $h_\varepsilon : M \to M$. The maps $\psi_\varepsilon = \phi o h_\varepsilon$ have the same properties as $\phi$ and only differ from $\phi$ on a small set. Thus, if $h = \sigma^2(w)(du^2 + dv^2)$ in some chart and $\psi_\varepsilon$ is represented by $z \to w_\varepsilon$, $\phi$ by $z \to w$, then

$$E(\psi_\varepsilon) - E(\phi) = \int_{|z|<1} \left(\left|\frac{\partial w_\varepsilon}{\partial z}\right|^2 + \left|\frac{\partial w_\varepsilon}{\partial \bar{z}}\right|^2\right)\sigma^2(w_\varepsilon)\, dxdy$$

$$- \int_{|z|<1} \left(\left|\frac{\partial w}{\partial z}\right|^2 + \left|\frac{\partial w}{\partial \bar{z}}\right|^2\right)\sigma^2(w) dxdy.$$

Using the definition of $\psi_\varepsilon$ and carrying out the differentiation

$$E(\psi_\varepsilon) - E(\phi) = \int_{|z|<1} \left(\left|\frac{\partial w}{\partial \zeta}\right|^2 + \left|\frac{\partial w}{\partial \bar{\zeta}}\right|^2\right)\left(|1 + \varepsilon\lambda_z|^2 + |\varepsilon\lambda_{\bar{z}}|^2\right)\sigma^2(w_\varepsilon) dxdy$$

$$+ 4\mathrm{Re}\ \varepsilon\int_{|z|<1} \frac{\partial w}{\partial \zeta}\ \frac{\partial \bar{w}}{\partial \zeta}\ \lambda_{\bar{z}}\ (1+\varepsilon\lambda_z)\sigma^2(w_\varepsilon) dxdy$$

$$- \int_{|z|<1} \left(\left|\frac{\partial w}{\partial z}\right|^2 + \left|\frac{\partial w}{\partial \bar{z}}\right|^2\right)\sigma^2(w) dxdy .$$

Changing variables and using $|\xi|_z{}^2 - |\xi_{\bar{z}}|^2 = |1 + \varepsilon \frac{\partial\lambda}{\partial z}|^2 - |\varepsilon \frac{\partial\lambda}{\partial \bar{z}}|^2$

yields

$$E(\psi_\varepsilon) - E(\phi) = 4\mathrm{Re}\ \varepsilon\int_{|\xi|<1} \frac{\partial w}{\partial\zeta}\ \frac{\partial\bar{w}}{\partial\zeta}\ \sigma^2(w).\ \frac{\partial\lambda}{\partial\bar{\zeta}}\ d\xi d\eta + O(|\varepsilon|^2)$$

where $\zeta = \xi + i\eta$. The minimising property of $\phi$ yields the equation

$$\int_{|\zeta|<1} \frac{\partial w}{\partial\zeta}\ \frac{\partial\bar{w}}{\partial\zeta}\ \sigma^2(w)\ \frac{\partial\lambda}{\partial\bar{\zeta}}\ d\xi d\eta = 0.$$

But $\lambda$ is arbitrary, so the function $\frac{\partial w}{\partial\zeta}\ \frac{\partial\bar{w}}{\partial\zeta}\ \sigma^2(w) = a(\zeta)$ satisfies the distributional equation $\frac{\partial a}{\partial\bar{\zeta}} = 0$, and by the well known lemma of Weyl [ 1] is holomorphic. There remains to note that in these coordinates

$$(\phi^*h)^{2,0} = a(\zeta)d\zeta^2.$$

The basic question to be answered now is the following: if $\phi$ is S-harmonic is it necessarily harmonic? The answer is yes if $\phi$ is sufficiently regular.

5.15. PROPOSITION

If $\phi:M \to N,h$ is a $C^2$ S-harmonic map and $d\phi$ has rank 2 on a dense subset of M, then $\phi$ is harmonic.

Proof

This easy proof is adapted from [37]. First note that if z is a local parameter on M then $\phi$ is harmonic if and only if $\nabla_{\partial/\partial\bar{z}}\ \frac{\partial\phi}{\partial z} = 0$.

Now, $(\phi^*h)^{2,0} = h(\frac{\partial\phi}{\partial z}, \frac{\partial\phi}{\partial z})dz^2 = h\left(d\phi(\frac{\partial}{\partial z}), d\phi(\frac{\partial}{\partial z})\right)dz^2$

so it is holomorphic if and only if

$$0 = \frac{\partial}{\partial\bar{z}}\, h(\frac{\partial\phi}{\partial z}, \frac{\partial\phi}{\partial z}) = 2h(\frac{\partial\phi}{\partial z}, \nabla_{\partial/\partial\bar{z}} \frac{\partial\phi}{\partial z})$$

since $\frac{\partial\phi}{\partial z}$ is non-zero on a dense set, the continuous field $\nabla_{\partial/\partial\bar{z}} \frac{\partial\phi}{\partial z}$ must vanish identically so $\phi$ is harmonic.

For maps which are not so smooth less can, as yet, be said. In fact in the general situation all the author has been able to show is very little.

5.16. PROPOSITION

Suppose that $\phi:M \to (N,h)$ is a $C^1$ diffeomorphism, if $\phi$ is S-harmonic then it is harmonic.

Proof.

If $\psi_t:M \to N$ is a variation of the smooth map $\psi:M \to N$ supported in a chart, then on one hand

$$\frac{d}{dt} E(\psi_t)\Big|_{t=0} = \int_M \langle d\psi, \nabla v\rangle * 1 = -\int_M \langle\tau(\psi), v\rangle * 1. \qquad (12)$$

While on the other a direct calculation shows that if $(\phi^*h)^{2,0} = a(z)dz^2$ then

$$\tau(\phi)^{1,0} = \frac{-4}{\sigma^2(w)\rho^2(z)} \cdot \frac{w_{\bar{z}}a_{\bar{z}} - w_z\bar{a}_z}{|w_z|^2 - |w_{\bar{z}}|^2}$$

where $\rho^2(z)(dx^2 + dy^2)$ is a metric on M. Consequently,

$$\int_M \langle d\psi, \nabla v\rangle \; * \; 1 = 4\mathrm{Re} \int_M \left(\frac{w_{\bar z} a_{\bar z} - w_z \bar a_z)\bar v}{|w_z|^2 - |w_{\bar z}|^2}\right) dxdy \tag{13}$$

If, now $\psi_n$ is a sequence of smooth diffeomorphisms approximating $\phi$ in the $C^1$ topology, then

$$\int_M \langle d\phi, \nabla v\rangle \; * \; 1 = \lim_{n\to\infty} \int_M \langle d\psi_n, \nabla v\rangle \; * \; 1$$

$$= \lim_{n\to\infty} \; 4\mathrm{Re} \int_M \left(\frac{\partial w_n}{\partial \bar z} \cdot \frac{\partial a_n}{\partial \bar z} - \frac{\partial w_n}{\partial z} \cdot \frac{\partial \bar a_n}{\partial z}\right) \frac{\bar v}{J_n} \; dxdy$$

by (13), where $J_n = \left|\frac{\partial w_n}{\partial z}\right|^2 - \left|\frac{\partial w_n}{\partial \bar z}\right|^2$. Thus

$$\int_M \langle d\phi, \nabla v\rangle \; * \; 1 = 4\mathrm{Re}\int_M \frac{(w_{\bar z} a_{\bar z} - w_z \bar a_z}{|w_z|^2 - |w_{\bar z}|^2} \; \bar v \; dxdy \tag{14}$$

by the $C^1$ convergence $\psi_n \to \phi$. Since $\phi$ is S-harmonic $(\overline{a_{\bar z}}) = \bar a_z = 0$ so (14) reads

$$\int_M \langle d\phi, \nabla v\rangle \; * \; 1 = 0. \tag{15}$$

Since v was arbitrary (15) expresses the condition that $\phi$ is a critical point of E, by Proposition 1.5 it is harmonic.

<u>Remark</u>

The three properties of $\phi$ needed to make the proof work are:

(a) $\phi$ is continuous and has square integrable distributional derivatives.

(b) There is a suitable approximation $\psi_n \to \phi$. (16)

(c) The approximation

$$\mathrm{Re} \int \{\frac{\partial w_n}{\partial \bar{z}} (\frac{\partial a_n}{\partial \bar{z}}) - \frac{\partial w_n}{\partial z} (\frac{\partial \bar{a}_n}{\partial z})\} \frac{\bar{v}}{J_n} \, dxdy \to 0 \tag{17}$$

holds.

Clearly the proof will still work if the $\psi_n$ are less smooth but the approximations (16) and (17) still hold. In general it seems impossible to tell if such $\psi_n$ exist.

(e) An alternative approach

This section describes a method which, hopefully, will lead to a map satisfying the hypotheses of Proposition 5.14. The method is to perturb the integral E with a parameter $\varepsilon$, to solve the perturbed problem and let $\varepsilon$ tend to zero.

If $\phi \in H(M,N)$, i.e. it satisfies (i) - (iii) of Section (a), then Proposition 5.6 validates the calculation

$$\int_N (|z_w|^2 + |z_{\bar{w}}|^2)\rho^2(z)\,dudv = \int_M \frac{|w_z|^2 + |w_{\bar{z}}|^2}{|w_z|^2 - |w_{\bar{z}}|^2} \rho^2(z)\,dxdy \tag{18}$$

where $g = \rho^2(z)(dx^2 + dy^2)$ is any metric on M, thus $E(\phi^{-1}, g)$ may be regarded as an integral over M.

5.17. DEFINITION

For $\phi \in H(M,N)$ the perturbed energy $E_\varepsilon(\phi)$ is defined by

$$E_\varepsilon(\phi) = E(\phi) + \varepsilon E(\phi^{-1}, g),$$

where g is some arbitrary but fixed Riemannian metric on M.

5.18. PROPOSITION

For each $\varepsilon > 0$ the functional $E_\varepsilon$ attains its minimum on $H(M,N)$.

Proof

If $\phi_n$ is a minimising sequence for $E_\varepsilon$ then by (18) and Theorem 5.7 both $\phi_n$ and $\phi_n^{-1}$ form equicontinuous families. As before it may be assumed, by passing to a subsequence, that $\phi_n \to \phi$ and $\phi_n^{-1} \to \phi^{-1}$ for some $\phi$. By following the reasoning in Section (c) it can be seen that $\phi \in H(M,N)$. That $E_\varepsilon$ attains its minimum at $\phi$ follows because

$$E_\varepsilon(\phi) \le \liminf_{n \to \infty} E_\varepsilon(\phi_n).$$

Now set $a(z) = \sigma^2(w)w_z\bar{w}_z$, $k(z) = \dfrac{|w_z|^2 + |w_{\bar{z}}|^2}{|w_z|^2 - |w_{\bar{z}}|^2}$

and $\ell(z) = \dfrac{\rho^2(z)w_z\bar{w}_z}{|w_z|^2 - |w_{\bar{z}}|^2}$

in some system of local coordinates $|z| < 1$ on $M$ and $w$ on $N$. Notice that $k$ is a globally defined function on $M$ while $a$ and $\ell$ define quadratic differentials on $M$.

5.19. PROPOSITION

The minimising map of Proposition 5.18 satisfies

$$\int_{|z|<1} (a + \varepsilon\ell)\lambda_{\bar{z}}\, dxdy = \frac{\varepsilon}{2}\int_{|z|<1} k(\lambda.\rho^2)_z\, dxdy \tag{19}$$

for all $\lambda : U \to \mathbb{C}$ which are smooth with compact support.

Proof

Let z and $\zeta$ be related, as in the proof of 5.14, by $\zeta = z + t\lambda$ where t is a suitably small parameter. Then calculating the derivatives and changing variables as in 5.14

$$E(\psi_t^{-1},g) - E(\phi^{-1},g) = \int_{|z|<1} \frac{|w_\zeta|^2 + |w_{\bar\zeta}|^2}{|w_\zeta|^2 - |w_{\bar\zeta}|^2} \rho^2(\zeta)d\zeta d\eta$$

$$+ 4\mathrm{Re}\, t \int_M \ell(\zeta)\rho^2(\zeta)d\xi d\eta$$

$$- 2\mathrm{Re}\, t \int_M k(\zeta)(\lambda_\zeta \rho^2 + 2\rho_\zeta \rho\lambda)d\xi d\eta + O(|t|^2) .$$

Using the calculation made in the proof of 5.14, writing z for $\zeta$ and using the fact that $\phi$ is extremal for $E_\varepsilon$ gives (19).

Now for each $\varepsilon > 0$ let $\phi_\varepsilon$ be one of the minimising maps provided by 5.18. Set

$$\mu = \inf \{E(\phi);\ \phi \in H(M,N)\}.$$

5.20. PROPOSITION

The following approximation holds.

$$\lim_{\varepsilon\to 0} E_\varepsilon(\phi_\varepsilon) = \mu. \qquad (20)$$

Proof

For large $B > 0$ set $\mu_B = \inf \{E(\phi);\ \phi \in H(M,N), E(\phi^{-1},g) \le B\}$,

so that

$$\lim_{B\to\infty} \mu_B = \mu. \tag{21}$$

The methods of Section (c) show that there is a map $\phi_B \in (\phi \in H(M,N);\ E(\phi^{-1},g) \le B\}$ satisfying

$$E(\phi_B) = \mu_B . \tag{22}$$

Set $\varepsilon = 1/B^2$ then,

$$E_\varepsilon(\phi_\varepsilon) \le E_\varepsilon(\phi_B) = \mu_B + 1/B$$

so that as $B \to \infty$ , $\varepsilon \to 0$ and

$$\lim_{\varepsilon\to 0} E_\varepsilon(\phi_\varepsilon) \le \lim_{B\to\infty} \mu_B$$

so (21) together with the positivity of $\varepsilon$ yields

$$\mu \le \lim_{\varepsilon\to 0} E_\varepsilon(\phi_\varepsilon) \le \lim_{B\to\infty} \mu_B = \mu.$$

This is clearly sufficient to demonstrate (20).

5.21. <u>Corollary of proof</u>

Further, $\lim_{\varepsilon\to 0} \varepsilon E(\phi^{-1},g) = 0$ and $\lim_{\varepsilon\to 0} E(\phi_\varepsilon) = \mu$.

As a consequence of these results it can be seen that $\phi_\varepsilon$ provides a good approximation to a map minimising E in H(M,N) in that a suitable subsequence $\phi_{\varepsilon_i}$ converges uniformly to some $\phi: M \to N$ satisfying

$$E(\phi) \le \liminf_{\varepsilon \to 0} E(\phi_\varepsilon) = \mu. \tag{23}$$

The idea is to show that $\phi$ is S-harmonic by studying the convergence of $(\phi_\varepsilon * h)^{2,0}$.

To this end, let $a_\varepsilon(z)$, $k_\varepsilon(z)$ $\ell_\varepsilon(z)$ be related to $\phi_\varepsilon$ , as after the proof of 5.18, in some coordinate system $|z| < 1$. If $|\ell_\varepsilon|$ denotes the volume element associated to the quadratic differential defined by $\ell_\varepsilon$ then an application of the Schwarz inequality yields

$$\int_M |\ell_\varepsilon| \le \int_M k_\varepsilon * g\, 1 = E(\phi_\varepsilon^{-1}, g) \tag{24}$$

so consequently,

$$\lim_{\varepsilon \to 0} \varepsilon \int_M |\ell_\varepsilon| = \lim_{\varepsilon \to 0} \varepsilon \int_M k_\varepsilon * 1 = 0. \tag{25}$$

Now extend $a_\varepsilon$, $k_\varepsilon$, $\ell_\varepsilon$ to the whole of $\mathbb{C}$ by letting them be identically zero outside $|z| < 1$, the variational equation (19) reads

$$\int_{\mathbb{C}} (a_\varepsilon + \varepsilon \ell_\varepsilon)\lambda_{\bar{z}}\, dxdy = \frac{\varepsilon}{2}\int_{\mathbb{C}} (\lambda\rho^2)_z\, k_\varepsilon\, dxdy \tag{26}$$

for all smooth $\lambda$ with supp $\lambda \subset U$. Let $V: U \to [0,1]$ be smooth with compact support in U, with V(z) depending only on $|z|$ and satisfying $\int_{\mathbb{C}} V\, dxdy = 1$. If $0 < r_o < 1$ is chosen once and for all and $V_r(z) = \frac{1}{r^2} V(rz)$ for $0 < r < r_o$ then with * denoting convolution (26) reads

$$\int_{\mathbb{C}} (a_\varepsilon + \varepsilon\ell_\varepsilon)(\lambda_* V_r)_{\bar{z}}\, dxdy = \frac{\varepsilon}{2}\int_{\mathbb{C}} (\lambda.\rho^2 * V_r)_z k_\varepsilon \, dxdy \quad (27)$$

for all $0 < r < r_o$ and all smooth $\lambda: U \to \mathbb{C}$ with $\sup \lambda \subset \{z; |z| < 1 - r_o\} = D_{1-r_o}$. Using the well known properties of the convolution and using an integration by parts (27) becomes

$$\int_{\mathbb{C}} \frac{\partial}{\partial \bar{z}} ((a_\varepsilon + \varepsilon\ell_\varepsilon) * V_r)\, \lambda \, dxdy = \frac{\varepsilon}{2}\int_{\mathbb{C}} \frac{\partial}{\partial z}(k_\varepsilon * V_r)\lambda\rho^2 \, dxdy \quad (28)$$

again for all $0 < r < r_o$ and supp $\lambda \subset D_{1-r_o}$. Consequently

$$\frac{\partial}{\partial \bar{z}} ((a_\varepsilon + \varepsilon\ell_\varepsilon) * V_r) = \frac{\varepsilon}{2}\rho^2 \frac{\partial}{\partial z} (k_\varepsilon * V_r) \quad (29)$$

is an equality of smooth functions on $D_{1-r_o}$, and holds for all $0 < r < r_o$.

5.22. PROPOSITION

For every $z \in D_{1-r_o}$ and every disc $D \subset D_{1-r_o}$ with centre $z$,

$$2((a_\varepsilon + \varepsilon\ell_\varepsilon) * V_r)(z) - \frac{1}{\pi i}\int_{\partial D} \frac{((a_\varepsilon - \varepsilon\ell_\varepsilon) * V_r)(\zeta)d\zeta}{\zeta - z}$$

$$= -\frac{\varepsilon}{\pi}\int_D \frac{\rho^2(\zeta)\ \partial/\partial\zeta(k_\varepsilon * V_r)(\zeta)d\xi d\eta}{\zeta - z} \quad (30)$$

for every $0 < r < r_o$, where $\zeta = \xi + i\eta$.

Proof

This is just the standard Cauchy type representation theorem for $C^1$ functions with equation (29) taken into account.

Because the operator $\partial/\partial\bar{\zeta}$ is elliptic it turns out that the right hand side of (30) is very well behaved as $\varepsilon \to 0$. As a preliminary to the study of these quantities let $D^n f$ denote any $n^{th}$-order combination of derivatives of $f:\mathbb{C} \to \mathbb{C}$.

5.23. LEMMA

For each n and each $0 < r < r_0$ there exists a constant K, depending only on n and r such that

$$|D^n(k_\varepsilon * V_r)(z)| \leq KE(\phi_\varepsilon^{-1}, g) \tag{31}$$

for all $z \in \mathbb{C}$.

Proof

Suppose that $\rho^2(z) \geq L > 0$ on $D_1$, this can easily be arranged by altering z by a scaling factor if necessary. Then by well known properties of the convolution.

$$|D^n(k_\varepsilon * V_r)(z)| = |(k_\varepsilon * D^n V_r)(z)|$$

$$\leq \int_{\mathbb{C}} |k_\varepsilon(z) D^n V_r(z-\zeta)| \, d\xi d\eta$$

where $k_\varepsilon$ is the function identically zero outside $D_1$. If $\sup |D^n V_r| \leq LK$ then

$$|D^n(k_\varepsilon *V_r)(z)| \le LK\int_{\mathbb{C}} k_\varepsilon \,dxdy \le K\int_{\mathbb{C}} k_\varepsilon.\rho^2 \,dxdy$$
$$\le KE(\phi_\varepsilon^{-1},g).$$

Now fix n and $0 < r < r_o$ and note that the supports of $D^n(k_\varepsilon *V_r)$ are bounded independely of $\varepsilon$ so that, in view of (25), (31) shows that

$$\varepsilon D^n(k_\varepsilon *V_r) \to 0 \text{ uniformly and in } L^p \text{ for all } p > 1, \qquad (32)$$

this is the key fact which makes everything work.

5.24. DEFINITION

Suppose U is a bounded domain in $\mathbb{C}$ and $f:U \to \mathbb{C}$ is a function. If $0 < \alpha < 1$ say that f is *uniformly $\alpha$-Hölder continuous* on U if

$$\sup_{\substack{z_1 w\in U\\ z\neq w}} \frac{|f(z) - f(w)|}{|z - w|^\alpha} < \infty.$$

In this case call this number $|f|_{\alpha,D}$ and denote the class of such functions by $C^\alpha(D)$. The norm on $C^\alpha(D)$ is defined by

$$\|f\|_{\alpha,D} = \sup_D |f| + |f|_{\alpha,D}$$

with respect to which $C^\alpha(D)$ is a Banach space.

The necessary potential theory is contained in the following proposition.

5.25. PROPOSITION

For each n and each $0 < r < r_o$ the functions

$$K_\varepsilon(z) = \frac{\varepsilon}{\pi} \int_{\mathbb{C}} \frac{\rho^2(\zeta) D^n(k_\varepsilon * V_r)(\zeta)\, d\xi\, d\eta}{\zeta - z}$$

converge to zero in $C^\alpha(D)_{1-r_o})$.

Proof

The $K_\varepsilon(z)$ tend uniformly to zero on $D_{1-r_o}$ because

$$|K_\varepsilon(z)| \le \frac{\varepsilon}{\pi} \sup_{D_{1-r_o}} |D^n(k_\varepsilon * V_r)| \int_{D_{1+r_o}} \frac{1}{|\zeta - z|}\, d\xi\, d\eta$$

while this last term tends to zero by (32).

For the $|K_\varepsilon|_{\alpha,D}$ semi-norm, set

$$P_\varepsilon(z) = \frac{\varepsilon}{\pi} \int_{\mathbb{C}} D^n(k_\varepsilon * V_r)(\zeta)\left(\frac{1}{\zeta - z} - \frac{1}{z}\right) d\xi\, d\eta = K_\varepsilon(z) - K_\varepsilon(0).$$

Then

$$|P_\varepsilon(z)| \le \frac{|z|}{\pi} \, \|\varepsilon D^n(k_\varepsilon * V_r)\|_{L^p} \, \left\|\frac{1}{\zeta(\zeta - z)}\right\|_{L^q} \quad \text{for all } p > 2$$

by Hölder's inequality, with $\frac{1}{p} + \frac{1}{q} = 1$. But

$$\left(\int_{\mathbb{C}} |\zeta(\zeta - z)|^{-q} d\xi\, d\eta\right)^{1/q} = |z|^{2/q^{-2}} \left(\int |w(w-1)|^{-q}\, du\, dv\right)^{1/q}$$

$$= |z|^{2/q^{-2}}\, C_p \quad \text{for } z \neq 0$$

for some constant $C_p$ when $p > 2$. Consequently

$$|P_\varepsilon(z)| \leq |z|^{1-2/p} C_p \, \|\varepsilon D^n(k_\varepsilon * V_r)\|_{L^p} \quad \text{for } z \neq 0. \tag{33}$$

Now $P_\varepsilon(z_1-z_2) = K_\varepsilon(z_1) - K_\varepsilon(z_2)$ so (33) implies

$$|K_\varepsilon(z_1) - K_\varepsilon(z_2)| \leq |z_1 - z_2|^{1-2/p} C_p \, \|\varepsilon D^n(k_\varepsilon * V_r)\|_{L^p}$$

whenever $z_1 \neq z_2$ and $p > 2$. The result now follows in view of (32).

The result of this analysis is that as $\varepsilon \to 0$ the functions $\varepsilon(k_\varepsilon * V_r)$ tend uniformly to zero on $D_{1-r_o}$ together with all their derivatives. An exactly similar analysis applied to $\varepsilon(\ell_\varepsilon * V_r)$ shows the same result for these. Equation (30) shows that $a_\varepsilon * V_r$ converges in the same sense to a function $a_r$ on $D_{1-r_o}$.

5.26. PROPOSITION

The function $a_r$ is holomorphic in $D_{1-r_o}$ and is independent of $r \in (0, r_o)$. Consequently the subscript can be dropped.

Proof

From Equation (30) it is clear that $a_r$ satisfies

$$a_r(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{a_r(\zeta)d\zeta}{\zeta - z} \tag{34}$$

for every disc $D \subset D_{1-r_o}$ with centre $z \in D_{1-r_o}$, since $a_r$ is smooth it is necessarily holomorphic.

To see that $a_r$ is independent of $r$ note that by the properties of the convolution and the convergence of $a_\varepsilon * V_r$

$$\lim_{\varepsilon \to 0} a_\varepsilon * V_r * V_{r'} = a_r * V_{r'} \quad \text{in } D_{1-r_o} \tag{35}$$

while by Fubini's theorem

$$(a_\varepsilon * V_r * V_{r'})(w) = \int (a_\varepsilon * V_r)(\zeta+w)V_{r'}(\zeta)d\ d\xi d\eta$$

$$= \int\int a_\varepsilon(\zeta+w+z)V_r(z)V_{r'}(\zeta)d\zeta d\eta\ dxdy$$

$$= (a_\varepsilon * V_{r'} * V_r)(w).$$

So that

$$\lim_{\varepsilon\to 0} (a_\varepsilon * V_{r'} * V_r) = a_{r'} * V_r \quad \text{in } D_{1-r_o}. \tag{36}$$

Equations (35) and (36) show $a_r * V_{r'} = a_{r'} * V_r$ while the mean value property of holomorphic functions and the fact that $V_r(\zeta)$ depends only on $|\zeta|$ shows that

$$a_r(z) = \int a_r(z+\zeta)V_{r'}(\zeta)d\xi d\eta = (a_r * V_{r'})(z)$$

and

$$a_{r'}(z) = \int a_{r'}(z+\zeta)V_r(\zeta)d\xi d\eta = (a_{r'} * V_r)(z)$$

for $z \in D_{1-r_o}$. Thus $a_r$ and $a_{r'}$ agree and the result is proved.

This section is concluded by a series of remarks.

(a) As $\varepsilon \to 0$ the $\phi_\varepsilon$ are uniformly equicontinuous by 5.7 and so some subsequence $\phi_{\varepsilon_i}$ converges uniformly to a continuous map $\phi: M \to N$ as already noted.

(b) It can be shown, as in 5.9 that $\phi$ maps null sets to null sets and that if $\tilde{\phi}: \tilde{M} \to \tilde{N}$ is a lift of $\phi$ and $B \subset M$ is a Borel set

$$\int_{\phi(B)} dudv = \int_B \left(\left|\frac{\partial\tilde{\phi}}{\partial z}\right|^2 - \left|\frac{\partial\tilde{\phi}}{\partial\bar{z}}\right|^2\right)dxdy$$

and consequently the Jacobian of $\phi$ is positive almost everywhere.

(c) If $(\phi^*h)^{2,0} = b(z)dz^2$ in the local coordinate discussed above then $b(z) = a(z)$ in $D_{1-r_o}$ by the weak convergence of the derivatives of $\phi_{\varepsilon_i}$ to those of $\phi$. Thus $\phi$ is S-harmonic.

(d) If it were known that an S-harmonic map with the above regularity properties of $\phi$ was harmonic the problem would be solved. However this is unknown to the author, all he can show is that the derivatives of $\phi$ are locally in $L^p$ for all $\infty > p > 1$ in the case that genus (M) = genus (N) = 1.

In conclusion then, the paper of Shibata is not completely revived but a certain amount can be said. Shibata's method itself can be carried in a certain way, indeed the variational problems of section (c) can both be solved (theorems 5.10 and 5.12). Let $\phi_A$ be a solution provided by 5.12. The next step in this method of attack would be to show that $\phi_A$ is S-harmonic when the parameter A is sufficiently large but the author disbelieves Shibata's proof of this and is unable to provide one of his own. These difficulties led to the search for a new idea and the alternative approach was developed. The result of this work is, in view of the above remarks, to find an S-harmonic map in the right homotopy class which is in some way well behaved and is also the uniform limit of a good set of homeomorphisms. This is all that at present can be proved. It would be nice to be able to show that the map constructed is a homeomorphism and even better to show that it possesses some higher degree of differentiability. This would show the existence of a harmonic diffeomorphism homopotic to the given $\psi: M \to N$.

# APPENDIX

## HARMONIC MAPPINGS OF OPEN SURFACES

The question keeps arising of whether there is an analogue of the equidistribution theory of holomorphic maps for harmonic maps of surfaces which admit only a given type of singularity. The result of this section is that in one very special case the answer is in the affirmative. However, the method is very ad. hoc and lends no insight to the general case.

Consider a harmonic map $\phi:M \to (N,h)$ of Riemann surfaces and let $(x,y)$ be local isothermal coordinates on M and $(u,v)$ normal coordinates on N by a result of Wood [38] these coordinates can be so chosen that

$$u(x,y) = \mathrm{Re}(a_k z^k) + O(|z|^k)$$
$$v(x,y) = \mathrm{Re}(b_\ell z^\ell) + O(|z|^\ell)$$

where $z = x + iy$. According to Wood [loc.cit] the singularity of the map at the centre of the system $(x,y)$ is determined by the nature of $\ell,k$ and $\mathrm{Im}(a_k \bar{b}_\ell)$.

### A.1. DEFINITION

The map $\phi:M \to N$ has a *branch point* at $p \in M$ if there exist isothermal coordinate $(x,y)$ centred at p and normal coordinates $(u,v)$ centred at $\phi(p)$ such that

$$u(x,y) = \mathrm{Re}(a_k z^k) + O(|z|^k)$$
$$v(x,y) = \mathrm{Re}(b_k z^k) + O(|z|^k) \quad k > 1 \text{ and } \mathrm{Im}(a_k \bar{b}_k) \neq 0.$$

In the case $N = \mathbb{C}$ the coordinates $(u,v)$ can be obtained from standard coordinates by a linear map. Note that for such a map the Jacobian is of constant sign with isolated zero.

As a preliminary, and for the sake of completeness, the following theorem of Osserman is included. It can be found in a slightly different form in [24].

A.2. THEOREM

If $\phi:\mathbb{R}^2 \to \mathbb{R}^2$ is harmonic and is a local diffeomorphism then their exists a non-singular linear map $A:\mathbb{R}^2 \to \mathbb{R}^2$ such that the composition $A \circ \phi$ is holomorphic with respect to the usual complex structure on $\mathbb{R}^2$

Proof

With subscripts denoting differentiation and superscripts the components of maps,

$$\phi^1_z = \tfrac{1}{2}(\phi^1_x - i\phi^1_y), \ \phi^2_z = \tfrac{1}{2}(\phi^2_x - i\phi^2_y)$$

are the complex gradients of $\phi$. Since the harmonic equation takes the form

$$\Delta\phi^i = 4\phi^i_{z\bar{z}} = 0 \quad i = 1,2$$

these complex gradients are holomorphic functions, consequently so is the function $z \longrightarrow \dfrac{\phi^2_z}{\phi^1_z}$ . Notice that $\phi^1_z$ never vanishes by the topological restriction on $\phi$.

Moreover a simple calculation shows

$$\operatorname{Im} \frac{\phi^2_z}{\phi^1_z} = \frac{1}{|\phi^1_z|^2} \operatorname{Im} \ \bar{\phi}^1_z \ \phi^2_z = \frac{-J}{|\phi^1_z|^2}$$

where $J$ is the Jacobian of $\phi_1$. Consequently

$$\operatorname{Im}\frac{\phi_z^2}{\phi_z^1} < 0 .$$

(It has tacitly been assumed that $\phi$ is orientation preserving, this is clearly no further restriction on $\phi$). By Liouville's theorem $\frac{\phi_z^2}{\phi_z^1}$ is constant so there exist numbers $a \in \mathbb{R}$, $b > 0$ satisfying

$$\frac{\phi_z^2}{\phi_z^1} = a - ib.$$

Writing this equation out in real notation yields

$$\phi_x^2 = a\phi_x^1 - b\phi_y^1$$

$$\phi_y^2 = b\phi_x^1 - a\phi_y^1$$

which is on elliptic first order system of equations. If $A$ is the linear map with matrix $\begin{pmatrix} 1 & 0 \\ -a/b & 1/b \end{pmatrix}$ and $\psi = A \circ \phi$ then $\psi$ satisfies the system

$$\psi_x^1 = \psi_y^2$$

$$\psi_y^1 = -\psi_x^2$$

which says that $\psi$ is holomorphic.

Remark

By a theorem of H. Lewy [19] the map $\phi$ need only be assumed to be a local homeomorphism.

However, if a harmonic map can be written as the composition of a holomorphic mapping followed by a non-singular linear mapping it can only admit branch parts as singularities. The converse is also true.

A.3. Theorem

If $\phi:\mathbb{R}^2 \to \mathbb{R}^2$ is a harmonic map with only branch points as singularities then there is a non-singular linear transformation A of $\mathbb{R}^2$ such that the composition $A \circ \phi$ is holomorphic with respect to the usual complex structure.

Proof

The idea of the proof is to consider the holomorphic function $\frac{\phi_z^2}{\phi_z^1}$ and to show that all its singularities are removable. These singularities occur at the branch point of $\phi$. Let $z_o$ be a branch point, then there exists a linear transformation T, so that if $\psi = T \circ \phi$ then for all z in a neighbourhood of $z_o$

$$\psi^1(z) = u_o + \mathrm{Re}\, a(z-z_o)^k + O(|z-z_o|^k)$$

$$\psi^2(z) = v_o + \mathrm{Re}\, b(z-z_o)^k + O(|z-z_o|^k)$$

for some $k > 1$ with $\mathrm{Im}(a\bar{b}) \neq 0$, $\quad T \circ \phi(z_o) = u_o + iv_o$.

By direct calculation

$$\psi_z^1 = \frac{k}{2}\, a(z-z_o)^{k-1} + O(|z-z_o|^{k-1})$$

$$\psi_z^2 = \frac{k}{2}\, b(z-z_o)^{k-1} + O(|z-z_o|^{k-1})$$

while it is clear that it can be assumed that $J \geq 0$.

Consequently,

$$\lim_{z\to z_o} \frac{\psi_z^2}{\psi_z^1} = \lim_{z\to z_o} \frac{a(z-z_o)^{k-1} + O(|z-z_o|^{k-1})}{b(z-z_o)^{k-1} + O(|z-z_o|^{k-1})}$$

$$= \frac{a\bar{b}}{|b|^2}$$

and the, at least locally defined, holomorphic function $\frac{\psi_z^2}{\psi_z^1}$

has a removable singularity at $z_o$. Noting that if T has matrix $\begin{pmatrix} c & d \\ e & f \end{pmatrix}$ then

$$\phi_z^1 = c\psi_z^1 + d\psi_z^2$$

where defined

$$\phi_z^2 = e\psi_z^1 + f\psi_z^2$$

so that

$$\lim_{z\to z_o} \frac{\phi_z^2}{\phi_z^1} = \lim_{z\to z_o} \frac{c\psi_z^1 + d\psi_z^2}{e\psi_z^1 + f\psi_z^2}$$

$$= \lim_{z\to z_o} \frac{c + d\psi_z^2/\psi_z^1}{e + f\psi_z^2/\psi_z^1}$$

$$= \frac{|b|^2 c + da\bar{b}}{|b|^2 e + fa\bar{b}}$$

provided that $e \neq \frac{fa\bar{b}}{|b|^2}$, but this is true since $e, f \in \mathbb{R}$ and $\operatorname{Im}(a\bar{b}) \neq 0$. Thus $\phi_z^2/\phi_z^1$ has a removeable singularity at each branch point of $\phi$ and the proof is now completed in exactly the same way as in the previous theorem.

A.4. <u>COROLLARY</u>

If $\phi: \mathbb{R}^2 \to \mathbb{R}^2$ is a harmonic map with only branch points as singularities then

(i) If $\phi$ is injective it is affine

(ii) If $\phi$ omits only one point it factors through the exponential map

(iii) If $\phi$ omits more than one point it is constant.

<u>Proof</u>

In each case the idea is to use the factorization of $\phi: \mathbb{R}^2 \to \mathbb{R}^2$ given by A.3. together with the appropriate property of the holomorphic map $\psi: \mathbb{C} \to \mathbb{C}$.

If $\phi: \mathbb{R}^2 \to \mathbb{R}^2$ factors

$$\begin{tikzcd} & \mathbb{R}^2 \arrow[d, "A"] \\ \mathbb{R}^2 \arrow[ur, "\psi"] \arrow[r, dashed, "\phi"] & \mathbb{R}^2 \end{tikzcd}$$

with A non-singular linear and $\psi$ holomorphic then $\phi$ is injective if and only if $\psi$ is injective. This means that in complex notation $\psi(z) = cz + d$, consequently $\phi$ is affine. If $\phi$ omits only one point then by composing with a translation this point can be assumed to be 0. Thus $\psi$ omits 0 and it is well known that $\psi(z) = e^{g(z)}$ for some holomorphic g. The final part of the corollary follows from the Little Picard Theorem in a similar way.

As a consequence of these results a few facts can be gleaned about harmonic maps from parabolic Riemann surfaces into complete flat surfaces.

A.5. THEOREM.

Let P be a parabolic Riemann surface and N a complete flat Riemannian surface which is not simply connected. If $\phi : P \to N$ is a harmonic map satisfying $J_\phi \gneq 0$ then $\phi$ is surjective.

Proof

First recall that P is parabolic if its universal covering $\tilde{P}$ space is conformally equivalent to the complex plane. If $\tilde{N}$ is the universal cover of N then $\phi$ admits a lifting which can be described by a harmonic map $\tilde{\phi} : \mathbb{C} \to \tilde{N}$.

$$\begin{tikzcd} \mathbb{C} \arrow[r] \arrow[rr, bend left, "\tilde{\phi}"] & \tilde{P} \arrow[r] \arrow[d] & \tilde{N} \arrow[d] \\ & P \arrow[r, "\phi"] & N \end{tikzcd}$$

By a well known theorem from differential geometry, see [16,I], $\tilde{N}$ is isometric to the complex plane so corollary A.4 may be applied. Suppose $\phi : P \to N$ is not onto, since $\pi_1(N) \neq 0$ the map $\mathbb{C} \xrightarrow{\tilde{\phi}} \tilde{N} \longrightarrow \mathbb{C}$ is a harmonic map omitting more than one point, by the corollary it is constant, hence so is $\phi : P \to N$.

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